جامعة بيرزيت
BIRZEIT UNIVERSITY

Birzeit University
Faculty of Graduate Studies
Master Program in Mathematics

Generalized Game Theoretical Model with Multiple Types of Homogeneous Players

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M.Sc.Thesis
Birzeit University
Palestine
2022

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This thesis was defended successfully on October 17, 2022 and approved by the following committee members

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Birzeit
October, 2022

الإهداء

إلى والدي ـ رحمه الله

Acknowledgments

First and foremost, I would like to express my sincere gratitude to my supervisor Dr. Abdelrahim Mousa for his invaluable advice, guidance and continuous support throughout writing this thesis. Also I would like to thank the committee members: Dr. Mohammad Saleh and Dr. Ala Talahmeh for their valuable notes and insightful comments. I would also like to thank all faculty members in the Department of Mathematics at Birzeit University for being my second family during my study years.

I would be remiss if I do not mention the first supporter in my life, my departed father, who sacrificed for educating and preparing me for the future. I am extremely grateful to my mother for her endless love, prayers and caring, and to my siblings (Obaida, Amer, Somaia, Amal and Ahmad) for their continuous support. My special gratitude goes to my soulmate and life partner (Faris), who never stopped encouraging me to do my best and provided me with endless emotional support. Also to my sweet little son (Ahmad), who brings joy to my life even in the rough time. I would also like to express my deep thanks to my husband's family, especially my mother-in-law, who has been a huge supporter and helper. I also would like to thank my friends for their encouraging words.

Abstract

We introduce a generalized decision game model which consists of finite number of types and each type has finite number of homogeneous players. We assume that each type of player has similar characteristics and will choose only between two alternative choices (or decisions) $D = \{d_1, d_2\}$. The preference for each type of players will be described by a discrete utility function which gathers the influence of players in the same group and the influence of players from the other groups. We will characterize all pure and mixed strategies that form Nash equilibria. The pure strategies are either united or separated strategies. The united strategies ensure that all players with same type will make same decision, while separated strategy includes at least one type of players who do not make same decision. We will determine the strategic thresholds for each type that identify the Nash regions in space. As a special case, we consider a game theoretical model for three types of homogeneous players. We use geometry to construct three dimensional regions of Nash equilibria, where the horizontal axis reflects the preference for players of type one, the vertical axis reflects the preference for the players of type two, and the depth axis reflects the preferences for the players of type three. We will identify Nash equilibria for pure and mixed strategies. Finally, we will apply our model in economics, specifically in the tourist sector. We will introduce a resort model for three types of tourists and find the Nash Equilibrium prices, for given horizontal preference for type 1, vertical preference for type 2, and depth preference for type 3.

الملخص

نقدم في هذه الرسالة نموذج لعبة قرار معمم يتكون من عدد محدود من الأنواع ولكل نوع عدد محدود من اللاعبين المتجانسين. نفترض أن كل نوع من اللاعبين له خصائص متشابهة وسيختار فقط بين خيارين بديلين. سيتم وصف تفضيل كل نوع من اللاعبين من خلال وظيفة المنفعة المنفصلة والتي تجمع تأثير اللاعبين في نفس المجموعة وتأثير اللاعبين من المجموعات الأخرى. سوف نعمل على تصنيف جميع الاستراتيجيات النقية والمختلطة التي تشكل توازنات ناش. الاستراتيجيات النقية إما استراتيجيات موحدة أو منفصلة. تضمن الاستراتيجيات الموحدة أن يتخذ جميع اللاعبين من نفس النوع نفس القرار، بينما تتضمن الإستراتيجية المنفصلة نوعًا واحدًا على الأقل من اللاعبين الذين لا يتخذون نفس القرار. سوف نحدد العتبات الاستراتيجية لكل نوع، والتي من خلالها يتم تحديد مناطق ناش في الفضاء. كحالة خاصة، سوف نأخذ نموذج لعبة نظري لثلاثة أنواع من اللاعبين المتجانسين. سنستخدم الهندسة لبناء مناطق ثلاثية الأبعاد تضمن تواجد توازنات ناش، حيث يعكس المحور الأفقي فيها تفضيل اللاعبين من النوع الأول، بينما يعكس المحور العمودي فيها تفضيل اللاعبين من النوع الثاني، ويعكس محور العمق فيها تفضيلات اللاعبين من النوع الثالث. سنحدد مناطق توازن ناش للاستراتيجيات النقية والمختلطة. أخيرًا، سنطبق نموذجنا في الاقتصاد، وتحديداً في قطاع السياحة. سنقدم نموذج لعبة اقتصادي يتكون من

منتجعين بحيث يتواجد ثلاثة أنواع من السياح حيث سنجد الأسعار التي تشكل توازن ناش. على فرض أن تفضيلات الأنواع الثلاثة من السُيّاح معروفة.

Contents

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List of symbols

$\mathbb{R}$	Real numbers
$\mathbb{R}^n$	Space of n-dimension
NE	Nash Equilibruim
ND	Nash domain
$\boldsymbol{S}$	Space of all strategies
S_U	United strategy
$\overline{\boldsymbol{S}_U}$	Space of united strategies
$\boldsymbol{NR}$	Nash region
$\boldsymbol{NP}$	Nash price
S_D	Separated strategy
$\overline{\boldsymbol{S}_D}$	Space of separated strategies

Chapter 1

Introduction

Human action is an important field of study in science research (especially social and economic science), where the famous Professor of Psychology Ajzen has studied across years the human behaviour theories [1,2,3], which are considered the most important theories in human actions. Ajzen in 2002 (see [4]) has studied behavioural control, and clarified how the planned behavior theory became influential and popular conceptual frameworks for the study of human action. In this theory Ajzen clarify that players behaviour are driven by three main beliefs (or factors): first is their thoughts and expectations, second is their expectations about other's behaviour and finally is the expectations about external factors. Briefly, they show how intentions of human turn into behaviours.

Recently, game theory field has become interested topic to study more complex human behavior using mathematical models and equations, trying to expect the best decisions that people can make among different available and alternative choices.

One important application of game theory in real life concerning human decision models is in economics, mainly in tourism sector. In 2010, Brida et al. (see[5]) introduced a model of tourism choice taking into account the crowding types (the effect of tourists on the decision of each other-demand on resorts), and found the strategies that achieve Nash Equilibrium. Also, in the same sector, Brida et al. (see [6]) in 2011, they found that adjustments in the number of tourists of each crowding type and adjustments in the parameters that characterize the utility of tourists could in fact modify the Nash Equilibrium allocation. Also, in "Resort Pricing and Bankruptcy" paper (see [7]) the authors demonstrate that slight changes in tourist preferences, as well as the preferences of tourists who like to be with in each resort, can build and destroy competitive business Nash Equilibrium prices, as well as change bankruptcy Nash Equilibrium prices.

At the same year, Pinto et al. (see [8]) introduced Yes-No decision game for two types of people using decision tilings. Each decision tiling demonstrates the way united and separated Nash equilibria co-exist and alter with the relative choice preferences of the individuals for the yes or no decision model.

In conjunction with previous work, Almeida et al. [9] created a game theoretical model, inspired by Conley (see [10]) and Wooders [10,11], where it takes into consideration individual characteristics described as taste type and crowding type effects. Among a lot of mechanisms to transform human intentions into behaviour, the paper proposes the Bayesian–Nash equilibrium, and eventually shows what could lead to split strategies.

In 2014, Mousa et al. [12] introduced a dichotomous decision model where people has to choose between two decisions (Yes or No), where the choice of people affected by their preferences and people can influence each other (crowding type). In their paper, they took two types of people and clarify geometrically in two dimensions, where the preference of type one is presented on horizontal axis, and the preference of type two is presented on vertical axis. The authors found the domains of Nash Equilibruim (pure and mixed) and used geometrically the tilings to clarify the equilibria.

Another application for game theoretical model of two types of players in the formation of societies is studied in [13], where individuals are characterized according to their main components: personal evaluation component and an interaction component. Such components are affected by some externally factors on individuals behaviours.

Resorting to "dynamic of human decisions" paper [12], Mousa and Pinto (see [14]) studied two geometric approaches to construct all possible decisions tilings. They found the Nash domains for the pure and mixed strategies, and they characterize the space of all parameters where the pure Nash equilibria are either cohesive or disparate. Eventually, they added a new interest features about the occurrences of bifurcations between the horizontal and vertical thresholds.

After that, Mousa and Shoman (see [15]) studied in details the analysis of transition of Nash Equilibria for single type of homogeneous players and identified all pure and mixed Nash Equilibria.

In 2020, Mousa and Rajab (see [16]) added an interesting feature to Mousa model introduced in [12] by considering the envy behaviour effect within two types of players. They studied all pure envy strategies that form Nash Equilibria. Furthermore, they characterize geometrically the preferences for both types of players along the horizontal and vertical axes. At the same year Hou and You (see [17]) introduced an equilibrium definition for the group decision and examined its relationship with the Nash Equilibrium in game theory. The experts' preferences in the group decision situation are represented by preference sequence vectors (PSVs), and found that the introduced 'group decision equilibrium' is formally similar to the Nash Equilibrium described by using such PSV.

Another important application of game theory is in engineering and networks, where Jizhao and Maojiao (see [18]) studied Nash Equilibrium for different types of players interacting in a distributed network. They used two methods (algorithms): the first one is centralized algorithm where player can access its own gradient value, while the second one is a distributed algorithm where the players are assumed to have limited access into the other players' actions.

One of the recent papers which investigates in Game Theory is Dong paper (see [19]) which shows a comparison between classical game theory and evolutionary game theory which neither requires the participants to be completely rational and nor requires complete information. They show that classical game theory deals more with the relative benefits of individual strategy, where evolutionary game theory deals with strategy changes as evolutionary forces change within a population over longer time scales.

Recently, an attractive studies in applied game theory according to Covid-19 pandemic, where Manchanda et. al. (see [20]) tried to highlight and analyse measures to prevent the spread of the virus. They studied three games scenarios according to Covid-19 pandemic payoffs between countries choosing between lock-downs and not, Battle of the sexes where choosing between going out or stay at home between couples, and illustrates the vaccinated and unvaccinated people along with their utilities. The authors in this paper showed how the vaccination drive can help in combating the virus and secure most individuals from the risk of contamination. Lockdowns and other preventative measures, such as staying inside the home, can help in secure millions of people and keep them away from the risk of having Corona virus.

We organize this Thesis as follows. In Chapter 2 we will review some definitions which help us understanding basics and proceed in game theory in our thesis. In Chapter 3, we will introduce the general set up for the model, then we will study the pure united and pure separated strategies that form Nash equilibria and take a spacial case for three types of players. After that we study the mixed strategies. In Chapter 4 we apply our model in tourist sector and take in consideration two types of resorts and find the Nash Equilibrium prices. We get our conclusion in chapter 5. In the end, we have four appendices, where the three appendices (A, B and C) contains additional cases for given Theorems and Lemma, while in Appendix D, we present a simulation in a special case for three types of players using Mathematica.

Chapter 2

Basic Preliminaries

Our purpose in this chapter is to review some basics in game theory that will be needed through this thesis. For instance, the definition of games, types of games, and applications of game theory.

2.1 Definition of games

In this section, we specify how to define a game in game theory i.e. key ingredients. Any game has three important factors or elements [21] as the following

1. The set of players $i \in I$ which will be a finite set in this thesis. The decision makers can be people, government, etc $\cdots$.

2. Actions, what can the players do, or what actions can players actually take? For example bargaining action, decide whether or not to strike, or decide whether to fund strike. Or investing action, investor decide how much of a stock to buy or sell, decide when to sell a stock, how they should react to other people in market, how they should behave when prices change. These choices give us the strategy space $\boldsymbol{S}_i$ for each player $i \in \boldsymbol{I}$.

3. Payoff functions π_i that gives player i's Von Neuman-Morgenstern utility $\pi_i(S)$ based on the strategy $S \in \boldsymbol{S}_i$. This is what motivates players to select certain strategy and not the others.

2.2 Concepts in game theory

Referring to the textbook of Watson [22], some questions were raised. For example, in a given game:

- Does the winner or looser exist?
- Is there any kind of cooperation between players?
- Is the game considered to be competitive? aggressive?

See next example of a game where conflict and cooperation are considered.

Example 2.1. *[22] Contract problem*
Assume there are a contract problem between a worker and his employer. May be there is a need to negotiate a wage contract to produce an economical product. Despite a conflict between parties on wage, there are also common interests in other dimensions. In some situations, like exceptional achievements, both parties may prefer to conclude a bonus in the contract, which may give the worker an excellent incentive to make a profit.

Keep in mind that conflict and cooperation overlap as in Watson textbook [22] Game theory is a methodology of formally studying situations of interdependence. In this book the author explains game theory formally using a mathematically precise and logically consistent structure. With the proper theoretical tools in place, we can study behavior in various contexts and better understand economic and, more generally, social interaction.

As mentioned in [23], a game is played whenever people interact in different fields, including life situations. For example, a car driver playing a game with other car drivers on a busy city street. Also, Napoleon and Wellington were playing a game at the Battle of Waterloo.

Ken [23] clarify that game theory as a discipline began with the publication of Von Neumann and Morgenstern's book, The Theory of Games and Economic Behavior, in 1944. Game theory became more popular after "A Beautiful Mind" movie, which talks about the life of "John Nash", a mathematician famous who won the Nobel Prize in Economics and made fundamental changes in game theory since 1950's.

In most societies, people interact constantly where Watson [22] show that their interaction either cooperative interaction or competitive interaction (like two firms fighting for market share). And Ken [23] simplify how the game theory aims to investigate how rational people should interact when they have conflicting interests, i.e. it is mostly about what happens when people interact in a rational manner. Game theory is the mathematical modeling of strategic interaction among rational (and irrational) agents.

So, we can say in general, game theory is a way of thinking about strategic interactions between self-interested people. Self-interested can be measured by utility and preferences for a player who aims to maximize such expected or average utility.

2.3 Types of games

In this section we explain the types of games in game theory. Each type of games help to analyze various kind of problems or games.

2.3.1 Cooperative game

Definition 2.1. *[24] Cooperative game theory refers to an abstract and axiomatic analyses of bargains or behaviors that players might reach, without explicitly modeling the processes. The name "cooperative" derives in part from the fact that the analyses often incorporate coalitional considerations.*

As an example of cooperative game: a group of people together are caring out a certain project, so they cooperate between their decisions at every step.

The phrase "cooperative" might be ambiguous [25]. It does not imply that each agent is agreeable and will obey arbitrary instructions. Instead, it indicates that the fundamental modeling unit is the group rather than the individual agent. More exactly, in cooperative game theory we still describe the individual preference of agents, but not their probable actions. Instead, we have a coarser model of the capacities of distinct groups.

Example 2.2. *[25](Voting game)*
This is an example on cooperative game, let there are four political parties, pp_1, pp_2, pp_3, and pp_4, each with 45, 25, 15, and 15 representatives respectively, make up a parliament. A \$100 million spending bill is up for a vote, along with how much of it should be controlled by each party. Any legislation must receive a majority vote, or at least 51 votes, to pass; otherwise, each party gets zero to spend.

2.3.2 Non-cooperative game

Definition 2.2. *[24] Non-cooperative game theory describes models in which players' actions are directly modeled, assuming they will act selfishly. Watson [22] clarify that in this type, the framework treats every action taken by an agent as an independent action. A person decides on an individual behavior based on his or her preferences, independent of the other players in the strategic environment.*

In a non-cooperative game, the players decide on their own strategy to maximize their profit. As a result, non-cooperative game theory is more common and used than cooperative games. And our work in this thesis falls under this kind of game. Leyton [25] argued that the word "non-cooperative" might be deceptive as it may infer that the theory applies primarily to circumstances in which the interests of distinct agents conflict.

Leyton [25] also simplify the essential difference between the types is that the basic modeling unit in non-cooperative game theory is the individual (including his beliefs, preferences, and possible actions). In contrast, the group is the primary modeling unit in coalitional game theory.

Jack [24] clarify that the theory of non-cooperative games is more fundamental than that of cooperative games. It requests a detailed description of the game's rules so that the strategies available to the players can be studied in depth. The goal is then to identify a suitable pair of equilibrium strategies to label as the game's solution. On the other hand, cooperative game theory takes a more free-wheeling attitude. It concerns situations in which players can agree on what to do in the game before it begins. Furthermore, it is assumed that these negotiations will end with the signing of a binding agreement. It is argued that the precise strategies available in the game will not matter very much under these conditions. The game's preference structure matters because it determines which contracts are feasible.

In general, one have to be careful in considering a model that describes the game; where if the model is too simple, it may miss crucial parts of the actual games that one want to study. And if the model is overly sophisticated, it may obstruct analytics by overshadowing key factors.

We will represent in the following the two most important forms (or representative) of Non-cooperative game, which are the extensive form and the strategic (or normal) form.

2.3.2.1 Normal form

Definition 2.3. *[26] The strategic form is a simpler way to represent a game. To describe a game in strategic form, we only need to identify the set of players in the game, the set of options accessible to each player, and how players' payoffs are determined by the alternatives they decide.*

So, this kind of representation (which called also strategic form) consist of (as it is described in [25])

- A finite set of players say $\boldsymbol{I} = \{1, 2, \cdots, n\}$, and a player $i \in \boldsymbol{I}$.
- Strategy space $\boldsymbol{S} = (S_1 \times S_2 \times \cdots \times S_n)$, where S_i is a finite set of actions (strategies) available to player i. Each vector. $s = (s_1, s_2, \cdots, s_n) \in \boldsymbol{S}$ is

called an action profile (strategy).

- $\pi = (\pi_1, \pi_2, \cdots, \pi_n)$, where $\pi_i : \boldsymbol{S} \longrightarrow \mathbb{R}$ is payoff (a real-valued utility) function for player i. All players aims to choose a strategy that maximize their utilities.

Normal (strategic) form represented usually by a table. One of the most common example for non-cooperative game in strategic form is the "Prisoners' dilemma". See the following example.

Example 2.3. *Prisoner's dilemma [24]*

This is a very common and simple example of representation a game in normal form. Here the players are two prisoners. The game is presented in Table 2.1, each player has to choose between cooperate (C) or defect (D). The first entry indicates the payoff to player 1 as a function of the pair of actions, while the second entry is the payoff for player 2.

TABLE 2.1: A Prisoners' Dilemma Game.

Player 1 \ Player 2	C	D
C	-1,-1	-3,0
D	0,-3	-2,-2

The typical explanation for the payoffs in the prisoners' dilemma is as follows. The two players committed a crime and are now being held in different rooms in a police station. The prosecutor has approached each of them and informed them individually: "I will let you go if you confess and agree to testify against the other player, and the other player does not confess. If you both confess, I shall sentence you both to two years in prison. If you do not confess and the other player does, you will be found guilty and sentenced to three years in jail. If no one confesses, I will charge you with a lesser crime for which we have enough evidence to prosecute you, and you will each serve a year in jail." As a result, the payoffs in the matrix indicate time spent in prison in terms of years. The phrase cooperate refers to cooperating with the other player. The phrase defect relates to admitting and agreeing to testify, which violates the (implicit) agreement with the other player.

In this game one can see that the Nash Equilibrium strategy for both players is (D, D). *Since the best response for p1 if p2 selects C is D, the best response for p1 if p2 selects D is D, and the best response for p2 if p1 selects C is D, the best response for p2 if p1 selects D is D. Note also that in this game the Nash Equilibrium strategy* (D, D) *is not the best for both players since* (C, C) *dominates* (D, D). *However, each player does not know or he/she is not sure if the other player will select C, so they end up playing* (D, D) *although it is bad strategy for both.*

The other form of representing a non-cooperative game is the extensive form, which representing as a tree.

2.3.2.2 Extensive form (or Tree form)

In this kind of forms, we pay attention for two additional factors which are timing and information. We have here a set of players who select their strategies sequentially to form a tree, this tree consists of nodes and branches. The payoffs for each player representing at each end node.

Definition 2.4. *The extensive form is the most richly structured way to describe game situations as mentioned in Reger book [26]. Rebert [27] clarified that this form details the game's players, when each player has the opportunity to make a move, what each player can do at that time, what each player is aware of at that time, and the payoff each player will get for each possible move combination. Informally, Leyton [25] defined a perfect-information game as a tree in the sense of graph theory, where each node represents one of the players' option, each edge represents a potential action, and the leaves represent ultimate outcomes over which each player has utility or preference.*

Note that the game can be

1. Game with perfect information which means that each player observes the strategy being selected by the other player. In this kind of games all branches are solid and complete.

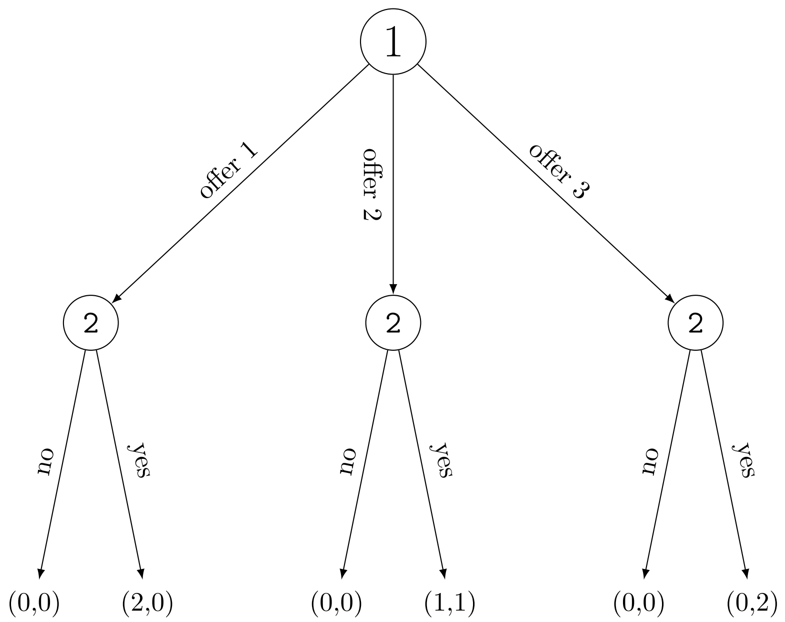


FIGURE 2.1: The sharing game.

2. Game with imperfect information which means that at least one of players does not observe what the other player has played previously. Such imperfect information games are represented by dash line in the tree.

The extensive form representation of a game specifies by Robert [27]

- The set of players in the game: the player who plays first starts at initial node.
- Strategy space for each player described by the branches.
- The payoff received by each player appear in the end of the tree.

Example 2.4. *[25] Suppose two siblings, a brother and a sister, are tasked with dividing up two identical gifts from their parents. Initially, the brother proposes a three-way division: either he retains both, she retains both, or they both retain one. The sister may then decide whether or not to accept the separation. If she agrees, they will get their respective gift(s); if not, neither will get anything. The game's tree structure is shown in Figure 2.1 under the assumption that the two gifts are of equal and additive value to each sibling.*

Note that the strategy space for player one in example 2.4 is

$$S_1 = \{\text{offer } 1, \text{offer } 2, \text{offer } 3\},$$

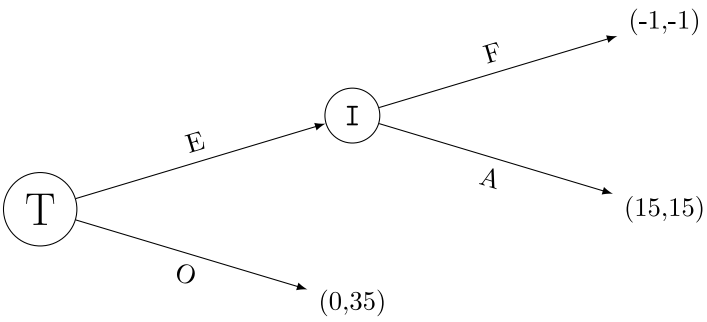


FIGURE 2.2: A basic entry deterrence game.

while the strategy space for player two

$$S_2 = \{yes, no\}.$$

In this example, we know that player one plays first and player two plays next after observing player's one action. Therefore, this is an example with complete information. Note also that this is an example with more than one Nash Equilibrium. That is (offer 1, yes), (offer 2, yes) and (offer 3, yes) are all Nash equilibria.

See the following extensive form as another example of non-cooperative game.

Example 2.5. *Companies fighting [28]*
In this example, we have two players (companies), see Figure 2.2. First is a potential industry entrant and take the symbol T and the second is industry incumbent take the symbol I. First company has two choices (actions), either to enter the industry (E) or stay out (O). After industry entrant choose, it became incumbent turn. If company T decide to stay out, there is no decision for company I, since the game terminated, and the payoffs for these companies are (0,35) for company T and I, respectively. If firm T decides to enter the market, then firm I has to choose between fighting (F)-launch a price war, which gives (-1,-1) payoffs for company T and I, respectively. Or, it decide to accommodate (A)- don't launch a price war, which gives (15,15) payoffs for company T and I, respectively. So, as we saw in this example, the decision of second player depends on the history of information taken by company one, which appear clearly how it depends on timing and information.

Note that from Example 2.5, the strategy space for the first company is

$$\{E, O\},$$

while the strategy space for the second company is

$$\{F, A\}.$$

In this market, the first company knows that if it decides E then the second company will decides A since A offers more payoff than F. Hence the first company ends having payoff 15 by deciding E, while if it decides O, it will get nothing. Therefore, the first company keeps deciding E and the second company keeps deciding A and the strategy profile (E, A) is the only Nash Equilibrium.

2.4 Strategy space

In game theory, strategy is an essential notion that may be defined as follows.

Definition 2.5. *[22] A strategy is a complete contingent plan for a player in the game.*

This is was also defined in Robert's work [27]. A player's strategy is a comprehensive action plan that outlines a possible action for the player in any situation in which the player may be required to perform..
Watson [22] clarified the concept of "complete contingent" in the preceding definition by providing a detailed description of a player's conduct at each of his potential choice points. A player's strategy specifies what he will do at each of his data, since data sets represent positions in the game when players make choices.

One kind of strategy is to choose a single action and play it with certainty. We refer to such a strategy as a pure strategy see [25], and will describe it using the notation we've previously defined for actions. According to Jack [24], the word "pure" refers to strategies in which a single action is chosen, as opposed to "mixed" strategies in which actions are randomized.

2.5 Best response

Another important concept in game theory is the player's belief about other players, where the players think about choices and strategies for each other player in the game. So, we can define a belief as following.

Definition 2.6. *[21] ΔS_{-i} is the set of probability distributions over the strategies of all players except player i.*

Definition 2.7. *[21] A belief of player $i \in \boldsymbol{I}$ is a probability distribution $\theta_{-i} \in \Delta S_{-i}$ over the strategies of the other players.*

Now we will define another kind of strategies (rather than the pure strategy), which called "mixed strategy".

Definition 2.8. *(Mixed strategy) [22]*
A mixed strategy for a player i is the act of selecting a strategy according to a probability distribution. A mixed strategy of player i can be denoted by $\sigma_i \in \Delta S_i$.

If player i has a belief θ_{-i} about the strategies of the others and decide to select strategy s_i, then his expected payoff is

$$\pi_i(s_i, \theta_{-i}) = \sum_{s_{-i} \in S_{-i}} \theta_{-i}(s_{-i}) \pi_i(s_i, s_{-i}).$$

So, we can say that a mixed strategy and a belief are both probability distributions. The difference underlines in the fact that the mixed strategy for player $i \in \boldsymbol{I}$ is a probability distribution over his/her strategy space, while the belief of player $i \in \boldsymbol{I}$ is probability distribution over the strategy space of the other players. Also, note that the set of mixed strategies includes the set of pure strategies, i.e. a pure strategy is a special case of a mixed strategy.

Definition 2.9. *[25] Player i's best response to the strategy profile s_{-i} played by the other players is a mixed strategy $s_i^* \in S_i$ such that $\pi_i(s_i^*, s_{-i}) \geq \pi_i(s_i, s_{-i})$ for all strategies $s_i \in S_i$.*

Best response isn't always unique. When a best response s^* involves two or more actions, the agent must be indifferent between them; otherwise, the agent would prefer to eliminate at least one action. Any blend of these acts is a best response, not only s^*. If two pure strategies are individually best responses, any blend of them is also best response. A player won't know the other players' strategy. As a result, the notion of best response is not a solution concept—it does not identify an interesting set of outcomes in this general case. However, we may use the concept of best response to establish what is probably the most essential concept in non-cooperative game theory: the Nash equilibrium.

2.6 Dominance

[25] We shall identify what it means for one strategy to dominate another. Intuitively, one strategy dominates another for a player i if the first strategy gives i a larger payoff than the second strategy, regardless of how the other players are playing. There are, however, three different levels of dominance, which are described in the following definitions.

Definition 2.10. *[25] Let s_i and s'_i be two strategies of player i, and S_{-i} the set of all strategy profiles of the remaining players. Then*

$$s_i \text{ strictly dominates } s'_i \text{ if for all } s_{-i} \in S_{-i}, \text{ we have}$$
$$\pi_i(s_i, s_{-i}) > \pi_i(s'_i, s_{-i}).$$

If one strategy dominates all others, we say that it is strongly dominant.

Definition 2.11. *[25] A player's strategy is strictly dominant if strictly dominates all other strategies for that player, and call it dominant strategy.*

Keep in mind that a strategy profile $(s_1, \cdots, s_n)$ is a Nash equilibrium if every s_i is dominant for player i. This kind of strategy profile is called an equilibrium in dominant (strictly) strategies. When there are strictly dominant strategies, then the equilibrium is the only possible Nash equilibrium.

Take, for instance, another look at the game known as "Prisoner's Dilemma" (see

Table 2.1). Whether player 1 chooses to play strategy C or D, player 2 will choose to play strategy D. And whether player 2 chooses to play strategy C or D, player 1 will choose to play strategy D. Therefore, the strategy D is strictly dominant for each player, and indeed (D, D) is the only Nash equilibrium.

In game theory, games with dominant strategies play a significant role. Next we will define the dominated strategy.

Definition 2.12. *[25] A strategy s_i for player $i \in \boldsymbol{I}$ is strictly dominated if some other strategy s_i' strictly dominates s_i.*

All strictly dominated pure strategies should be avoided, since they can never be best responses to what the other players do. When a pure strategy is eliminated, a different strategy that was not dominated can become dominated. So, this process of elimination can go on. Then, a pure strategy might be dominated by a combination of other pure strategies without being dominated independently by any of them. Take a look at the game in Table 2.2 for further clarification.

TABLE 2.2: A game with dominated strategies.

Player 1 \ Player 2	L	C	R
U	3,1	0,1	0,0
M	1,1	1,1	5,0
D	0,1	4,1	0,0

Column R, which is dominated by column L, can be eliminated for player 2. Table 2.3 shows the simplified version of the game that remains.

TABLE 2.3: Reduced game in Table 2.2 after removing the dominated strategy R.

Player 1 \ Player 2	L	C
U	3,1	0,1
M	1,1	1,1
D	0,1	4,1

Neither U nor D can dominates M in this game, but the mixed strategy that chooses either U or D with equal probability does. (Remember that it was not dominated before the R column was eliminated). This leaves us with the most simplified game shown in Table 2.4.

Table 2.4: Reduced game in Table 2.3 after removing the dominated strategy M.

Player 1 \ Player 2	L	C
U	3,1	0,1
D	0,1	4,1

This provides us with a solution concept: the set of all strategy profiles, each of which assigns zero probability to choose any action that would be eliminated via iterative elimination of strictly dominated strategies. Take into account that the strength of this solution notion is substantially weaker than than Nash equilibrium. Several many other mixed strategies might be part of the solution.

2.7 Nash Equilibrium

Definition 2.13. *[24]* ***Nash Equilibruim strategy***

A Nash equilibrium strategy is a profile of strategies such that each player's strategy is a best response (results in the highest available payoff) against all other strategies of the other players. That is, a strategy a_i is a best response of player i if

$$\pi_i(a_i, a_{-i}) \geq \pi_i(a'_i, a_{-i}).$$

Example 2.6. *Consider the strategic form game represented in Table 2.5. First,*

Table 2.5: A strategic game.

Player 1 \ Player 2	X	Y	Z
U	2,0	1,1	4,2
M	3,4	1,2	2,3
D	1,3	0,2	3,0

we can see that strategy D for player 1 is dominated by strategy U. And hence, strategy Y will be dominated by strategy Z. Therefore, the set $\{(U, X), (U, Z), (M, X), (M, Z)\}$ contains the rationalizable strategies.

To find NE strategies, we need to find best response for each player. If player 2 choose action X, then the best response for player 1 is M, since it has the highest pay off with 3. If player 2 choose action Y, then the best response for player 1 is U and M, since they have the same highest pay off with 1. If player 2 choose *action Z, then the best response for player 1 is U, since it has the highest pay off with 4.*

Now, we need to find best response for player 2 with respect to each action of player 1. If player 1 choose U, then the best response for player 2 is Z, since it has the highest payoff with 2. If player 1 choose M, then the best response for player 2 is X, since it has the highest payoff with 4. If player 1 choose strategy D, then the best response for player 2 is X, since it has the highest payoff with 3.

So, the Nash equilibria strategies are (U, Z) *and* (M, X) *for this game. Whereas the strategies* $\{(U, X), (M, Z)\}$ *are rationalizable strategies but not Nash equilibria.*

Chapter 3

Decision game model with interactions between multiple types of players

In this chapter, we extend work done by Mousa et al. introduced in [12] by generalizing a dichotomous decision model for multiple type of homogeneous player, who can choose only between two alternatives choices. We will study pure (united and separated) and mixed strategies that form Nash equilibria.

3.1 Model setup

In this section, we will formulate general game theory decision model for multiple type of homogeneous player, who can choose only between two alternatives choices. We will study pure (united and separated) and mixed strategies that form Nash equilibria.

Definition 3.1. *A group of m-players in any games are called homogeneous, if they have the same characteristics and the same preferences.*

Let $\boldsymbol{T} = \{t_1, t_2, \cdots, t_n\}$ be set with n−types of homogeneous players. Each type t_k has m_k players, where $k \in \{1, \cdots, n\}$.

Let:

$$\begin{aligned}
\boldsymbol{I_1} &= \{1, \cdots, m_1\} \text{ be the set of all players with type } t_1. \\
\boldsymbol{I_2} &= \{1, \cdots, m_2\} \text{ be the set of all players with type } t_2. \\
&\vdots \\
\boldsymbol{I_n} &= \{1, \cdots, m_n\}\text{be the set of all players with type } t_n.
\end{aligned}$$

Definition 3.2. *The disjoint union of* $\boldsymbol{I_1}, \boldsymbol{I_2}, \cdots, \boldsymbol{I_n}$ *is defined by*

$$\boldsymbol{I} = \boldsymbol{I_1} \sqcup \boldsymbol{I_2} \sqcup \cdots \sqcup \boldsymbol{I_n},$$

where

$$\boldsymbol{I} = \{i = (i_1, i_2, \cdots, i_n) \in \mathbb{R}^n : i_k \in \boldsymbol{I_k}, \;\; k \in \{1, \cdots, n\}\}. \tag{3.1}$$

Each player $i_k \in \boldsymbol{I_k}$ is assumed to make one decision $d \in \boldsymbol{D} = \{d_1, d_2\}$.

Let $\boldsymbol{\Omega_d} \in \mathbb{R}^n$ be the preference decision vector whose coordinates ω_k^d indicate how much player with type t_k likes or dislikes to make decision $d \in \boldsymbol{D}$. So, $\boldsymbol{\Omega_d}$ indicates the taste type of players defined by

$$\boldsymbol{\Omega_d} = \begin{pmatrix} \omega_1^d \\ \omega_2^d \\ \vdots \\ \omega_n^d \end{pmatrix} \in \mathbb{R}^n, \;\; d \in \{d_1, d_2\}.$$

Since the game consists of homogeneous players, it follows that ω_k^d is the same for all players of the same type t_k where $k \in \{1, \cdots, n\}$.

Let $\boldsymbol{A_d} \in \mathbb{R}^{n\times n}$ be the preference neighbors matrix (crowding matrix) whose coordinates $\alpha^d_{kj} \in \mathbb{R}$ indicate how much player with type t_k likes or dislikes that player with type t_j makes decision $d \in \boldsymbol{D}$. So, $\boldsymbol{A_d}$ indicates crowding type of players defined by

$$\boldsymbol{A_d} = \begin{pmatrix} \alpha^d_{11} & \alpha^d_{12} & \cdots & \alpha^d_{1n} \\ \alpha^d_{21} & \alpha^d_{22} & \cdots & \alpha^d_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha^d_{n1} & \alpha^d_{n2} & \cdots & \alpha^d_{nn} \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Let $\boldsymbol{S}$ be the space of all strategies $S \in \boldsymbol{S}$, where S is defined by

$$S = (S_1, S_2, \cdots, S_{k-1}, S_k, S_{k+1}, \cdots, S_n), \tag{3.2}$$

where the pure decision of the players of type t_k is described by a pure strategy map

$$S_k : \boldsymbol{I_k} \longrightarrow \boldsymbol{D}, \quad k = 1, 2, \cdots, n, \tag{3.3}$$

that associates to each player $i_k \in \boldsymbol{I_k}$ its decision $S(i_k) \in \boldsymbol{D}$.

Given a strategy S, let $\boldsymbol{O_S}$ be the strategic decision vector in $\mathbb{R}^n$ whose coordinates $l^d_k = l^d_k(S)$ indicate the number of players with type t_k who make decision d. So, $\boldsymbol{O_S}$ is defined by:

$$\boldsymbol{O_S} = \begin{pmatrix} l^{d_1}_1 \\ l^{d_1}_2 \\ \vdots \\ l^{d_1}_n \end{pmatrix} \in \mathbb{R}^n$$

We denote by $(l_1, l_2, \cdots, l_n) = (l^{d_1}_1(S), l^{d_1}_2(S), \cdots, l^{d_1}_n(S))$ the strategic decision vector associated with strategy S, where l_k (resp. $m_k - l_k$) is the number of player with type t_k who makes decision d_1 (resp. d_2) where $k \in \{1, \cdots, n\}$. That is:

$$\boldsymbol{O_S} = \begin{pmatrix} l_1^{d_1} \\ l_2^{d_1} \\ \vdots \\ l_n^{d_1} \end{pmatrix} = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{pmatrix} \quad \text{and so} \begin{pmatrix} l_1^{d_2} \\ l_2^{d_2} \\ \vdots \\ l_n^{d_2} \end{pmatrix} = \begin{pmatrix} m_1 - l_1 \\ m_2 - l_2 \\ \vdots \\ m_n - l_n \end{pmatrix}.$$

Lemma 3.3. *The number of all possible pure strategies is*

$$\prod_{k=1}^{n}(m_k + 1),$$

where n is the number of types and m_k is the number of homogeneous players of type t_k.

Proof. Since the model has n homogeneous types and each type t_k has m_k players, it follows that the set of all possible strategic decision vector is given by the Cartesian product of sets:

$$\boldsymbol{O} = \{0, 1, \cdots, m_1\} \times \{0, 1, \cdots, m_2\} \times \cdots \times \{0, 1, \cdots, m_n\}. \tag{3.4}$$

Hence, the number of all possible pure strategies now is the combination

$$(m_1 + 1).(m_2 + 1). \cdots .(m_n + 1) = \prod_{k=1}^{n}(m_k + 1),$$

which complete the proof. □

Definition 3.4. *Let $\epsilon_k(d)$ be the variation in the utility of players of type k who makes decision $d \in \boldsymbol{D}$.*

Now, we need to define the corresponding utility functions. Let $\Pi_k : \boldsymbol{D} \times \boldsymbol{O} \longrightarrow \mathbb{R}$ be the utility function of any player with type t_k who make decision d_1 is defined by:

$$\begin{aligned} \Pi_k(d_1; l_1, l_2, \cdots, l_n) &= \omega_k^{d_1} + \alpha_{k1}^{d_1} l_1 + \alpha_{k2}^{d_1} l_2 + \cdots + \alpha_{kk-1}^{d_1} l_{k-1} + \alpha_{kk}^{d_1}(l_k - 1) \\ &\quad + \alpha_{kk+1}^{d_1} l_{k+1} + \cdots + \alpha_{kn}^{d_1} l_n + \epsilon_k(d_1), \end{aligned} \tag{3.5}$$

and let $\Pi_k : \boldsymbol{D} \times \boldsymbol{O} \longrightarrow \mathbb{R}$ be the utility function of any player with type t_k who make decision d_2 is defined by:

$$\begin{aligned}\Pi_k(d_2; l_1, l_2, \cdots, l_n) &= \omega_k^{d_2} + \alpha_{k1}^{d_2}(m_1 - l_1) + \alpha_{k2}^{d_2}(m_2 - l_2) + \cdots \\ &\quad + \alpha_{kk-1}^{d_2}(m_{k-1} - l_{k-1}) + \alpha_{kk}^{d_2}(m_k - l_k - 1) \qquad (3.6) \\ &\quad + \alpha_{kk+1}^{d_2}(m_{k+1} - l_{k+1}) + \cdots + \alpha_{kn}^{d_2}(m_n - l_n) + \epsilon_k(d_2),\end{aligned}$$

where $k \in \{1, \cdots, n\}$ is the player type.

So, the utility is affected by two main factors: First, is the taste type represented by ω_k^d, which shows how much this player like to make decision d (i.e. the more a player desires to make decision d, the more utility the player will has related to this decision). Second is the crowding type represented by α_{kj}^d, which shows how much player of type t_k like or dislike to be with players of type t_j make decision d multiplies by the number of players whose made decision d.

Definition 3.5. *Let $x_k = \omega_k^{d_1} - \omega_k^{d_2}$ be the relative decision preference of the player with type t_k, where $k \in \{1, \cdots, n\}$.*

The relative decision preferences describe the taste preferences for each player independently from the influence of the other players.

- If $x_k > 0$, then player of type t_k prefers to decide d_1 without taking into account the influence of the others.
- If $x_k = 0$, then player of type t_k is indifferent to decide d_1 or d_2 without taking into account the influence of the others.
- If $x_k < 0$, then player of type t_k prefers to decide d_2 without taking into account the influence of the others.

Let

$$X = \{x \in \mathbb{R}^n : x = (x_1, x_2 \cdots, x_n)\},$$

be the set of all vectors of relative decision preferences.

Note that for a given strategy $S \in \boldsymbol{S}$, the utility function $\Pi_{i_k}(S)$ of player i_k with type t_k is given by

$$\Pi_k(S(i_k); l_1^{S(i_k)}(S), \cdots, l_n^{S(i_k)}(S)),$$

where $S(i_k) \in \{d_1, d_2\}$.

Definition 3.6. *Let $S^* = (S_1^*, S_2^*, \cdots, S_k^*, \cdots, S_n^*)$, where $S_k^* : \boldsymbol{I_k} \longrightarrow \boldsymbol{D}$. A strategy S^* is a (pure) Nash Equilibrium iff*

$$\Pi_{i_k}(S^*(i_k); l_1^{S^*(i_k)}(S^*), l_2^{S^*(i_k)}(S^*), \cdots, l_n^{S^*(i_k)}(S^*)) \geq$$
$$\Pi_{i_k}(S(i_k); l_1^{S(i_k)}(S), l_2^{S(i_k)}(S), \cdots, l_n^{S(i_k)}(S))$$

for every player $i_k \in I_k$ with type $k \in \{1, 2, \cdots, n\}$ and for every strategy $S \in \boldsymbol{S}$.

For every player $i_k \in \boldsymbol{I}_k$ and for every strategy $S \in \boldsymbol{S}$. The Nash region $\boldsymbol{NR}(S)$ of a strategy $S \in \boldsymbol{S}$ is the set of all preferences $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ for which S is a Nash Equilibrium.

3.2 United Nash equilibria

In this section, we will characterize the united strategies that are Nash equilibria and we will characterize its Nash equilibria regions.

Definition 3.7. *A united strategy $S_U = (S_1, S_2, \cdots, S_n)$ is a strategy in which all players with the same type prefer to make the same decision, where*

$$S_k : I_k \longrightarrow d_1 \text{ for all } i_k \in I_k \text{ or}$$
$$S_k : I_k \longrightarrow d_2 \text{ for all } i_k \in I_k,$$

i.e. $l_k \in \{0, m_k\}$ where $k \in \{1, \cdots, n\}$.

In other words, under the united strategy S_U, we have $S_U(i_k) = d_1$ for each player i with type t_k or $S_U(i_k) = d_2$ for each player i with type t_k.

Definition 3.8. *Let $\overline{\boldsymbol{S}_U^n}$ be the space of all united strategies S_U defined by the Cartesian product $\overline{\boldsymbol{S}_U^n} = \{0, m_1\} \times \{0, m_2\} \times \cdots \times \{0, m_n\}$.*

So, we can define $\boldsymbol{O}_{\boldsymbol{S_U}}$ as the strategic decision vector associated with united strategy S_U, by

$$\boldsymbol{O}_{\boldsymbol{S_U}} = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_n \end{pmatrix} \quad where l_k \in \{0, m_k\}. \tag{3.7}$$

Lemma 3.9. *The number of united strategies is 2^n, where n is the number of types.*

Proof. By Induction. Let $\overline{\boldsymbol{S}_U^n}$ be the space of all united strategies S_U.

$$\begin{aligned} If \;\; n = 1 \;\; &\Longrightarrow \;\; S_U(i_1) \in \{d_1, d_2\} \Longrightarrow l_1^d \in \{0, m_1\}, \;\; for \;\; d \in D\} \\ &\Longrightarrow \;\; \overline{\boldsymbol{S}_U^1} = \{0, m_1\} \\ &\Longrightarrow \;\; |\overline{\boldsymbol{S}_U^1}| = 2 = 2^1. \end{aligned}$$

Assume it is true for $n-1$[1] $\Longrightarrow |\overline{\boldsymbol{S}_U^{n-1}}| = 2^{n-1}$. We need to show it is true for n. Note that

$$\begin{aligned} \overline{\boldsymbol{S}_U^{n-1}} &= \{0, m_1\} \times \{0, m_2\} \times \cdots \times \{0, m_{n-1}\}, \\ &= \big\{(0, 0, \cdots, 0) \in \mathbb{R}^{n-1}, \cdots, (m_1, m_2, \cdots, m_{n-1}) \in \mathbb{R}^{n-1}\big\}, \end{aligned}$$

[1]See the appendix A for cases of two types and three types

with $|\overline{\boldsymbol{S}_U^{n-1}}| = 2^{n-1}$.

Now, let

$$\begin{array}{rcl} \overline{\boldsymbol{S}_U^{n}} & = & \overline{\boldsymbol{S}_U^{n-1}} \times \{0, m_n\}, \\ & = & \begin{array}{l} \{(0,0,\cdots,0,0) \in \mathbb{R}^n, \cdots, (m_1, m_2, \cdots, m_{n-1}, 0) \in \mathbb{R}^n, \\ (0,0,\cdots,0,m_n) \in \mathbb{R}^n, \cdots, (m_1, m_2, \cdots, m_{n-1}, m_n) \in \mathbb{R}^n\}. \end{array} \end{array} \quad (3.8)$$

Clearly $\overline{\boldsymbol{S}_U^{n-1}} = 2^{n-1} + 2^{n-1} = 2.2^{n-1} = 2^n$. □

We will represent a general form for Nash region and strategic thresholds of united strategies for n-types of players in two decision game.

Definition 3.10. *Given a united strategy S_U. Let*

N_1 be the set of all types of players t_k who decide d_1 ($l_k = m_k$),

N_2 be the set of all types of players t_k who decide d_2 ($l_k = 0$).

Note that under strategy S_U we have,

$$N_1 \cap N_2 = \phi \text{ and } \|N_1\| + \|N_2\| = n.$$

Given $S_U \in \overline{\boldsymbol{S}_U^{n}}$. The corresponding strategic decision vector is

$$\boldsymbol{O}_{\boldsymbol{S_U}}^{\boldsymbol{T}} = (l_1, l_2, \cdots, l_n) \in \mathbb{R}^n \text{ as given in (3.7).}$$

If players with type t_k make decision d_1, then this means

$$\Pi_k(d_1; l_1, l_2, \cdots, m_k, \cdots, l_n) \geq \Pi_k(d_2; l_1, l_2, \cdots, m_k - 1, \cdots, l_n), \quad (3.9)$$

while if this type make decision d_2, then this means

$$\Pi_k(d_2; l_1, l_2, \cdots, 0, \cdots, l_n) \geq \Pi_k(d_1; l_1, l_2, \cdots, 1, \cdots l_n). \quad (3.10)$$

If (3.9) and (3.10) hold for all types, then S_U is Nash Equilibrium.

Remark 3.11. *We must note that when we say given strategy $S_U \in \boldsymbol{S}$ is equivalent to say given a certain distribution of players over the two decisions. That is, given $S_U \in \boldsymbol{S}$, this means that the strategic decision vector $\boldsymbol{O^T_{S_U}} = (l_1, l_2, \cdots, l_n) \in \mathbb{R}^n$ is given. In this sense, sometimes instead of talking about S we may talk about $\boldsymbol{O^T_{S_U}}$ and we simply write*

$$S_U = (l_1, l_2, \cdots, l_n).$$

Theorem 3.12. *Given $S_U \in \overline{\boldsymbol{S}^n_U}$. The united strategy $S_U = (l_1, l_2, \cdots, l_n) \in \mathbb{R}^n$ is NE iff the following relative decision preference x_k of player with type t_k holds for all $k = 1, 2, \cdots, n$.*

$$x_k = \omega_k^{d_1} - \omega_k^{d_2} \geq -\alpha_{kk}^{d_1}(m_k - 1) - \sum_{j \in N_1, j \neq k} \alpha_{kj}^{d_1} m_j + \sum_{j \in N_2} \alpha_{kj}^{d_2} m_j + \epsilon_k(d_2) - \epsilon_k(d_1), \; k \in N_1,$$

$$x_k = \omega_k^{d_1} - \omega_k^{d_2} \leq -\sum_{j \in N_1} \alpha_{kj}^{d_1} m_j + \sum_{j \in N_2, j \neq k} \alpha_{kj}^{d_2} m_j + \alpha_{kk}^{d_2}(m_k - 1) + \epsilon_k(d_2) - \epsilon_k(d_1), \; k \in N_2,$$

where the sets N_1 and N_2 are as given in Definition 3.10.

Moreover the united strategic thresholds of S_U strategy are given by

$$\begin{aligned} X_k(l_1, \cdots, l_{k-1}, m_k, l_{k+1}, \cdots, l_n) &= -\alpha_{kk}^{d_1}(m_k - 1) - \sum_{j \in N_1, j \neq k} \alpha_{kj}^{d_1} m_j + \sum_{j \in N_2} \alpha_{kj}^{d_2} m_j \\ &\quad + \epsilon_k(d_2) - \epsilon_k(d_1), \qquad if \;\; k \in N_1, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} X_k(l_1, \cdots, l_{k-1}, 0, l_{k+1} \cdots, l_n) &= -\sum_{j \in N_1} \alpha_{kj}^{d_1} m_j + \sum_{j \in N_2, j \neq k} \alpha_{kj}^{d_2} m_j + \alpha_{kk}^{d_2}(m_k - 1) \\ &\quad + \epsilon_k(d_2) - \epsilon_k(d_1), \qquad if \;\; k \in N_2. \end{aligned} \tag{3.12}$$

Proof. By Induction. Given a united strategy $S_U = (l_1, l_2, \cdots, l_n) \in \mathbb{R}^n$.

$$If \;\; n = 1 \qquad \Longrightarrow \qquad S_U = (l_1) \;\; and \;\; l_1 \in \{0, m_1\}$$

$if \quad l_1 = m_1$ then the players of type t_1 make decision d_1 iff

$\Pi_1(d_1, m_1) \geq \Pi_1(d_2, m_1 - 1)$;

substituting the values of the utility from (3.5) and (3.6)

and rearrange terms, we get

$x_1 \geq -\alpha_{11}^{d_1}(m_1 - 1) + \epsilon_1(d_2) - \epsilon_1(d_1)$

using (3.11), one can show that

$-\alpha_{11}^{d_1}(m_1 - 1) + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(m_1)$

Hence, $x_1 \geq X_1(m_1)$.

and if $\quad l_1 = 0$ then the players of type t_1 make decision d_2 iff

$\Pi_1(d_2, 0) \geq \Pi_1(d_1, 1)$;

substituting the values of the utility from (3.5) and (3.6)

and rearrange terms, we get

$then \;\; x_1 \leq \alpha_{11}^{d_2}(m_1 - 1)$.

using (3.12), one can show that

$\alpha_{11}^{d_2}(m_1 - 1) + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(0)$.

Hence, $x_1 \leq X_1(0)$.

Assume it is true for $n - 1$[2], i.e. the united strategy $S_U = (l_1, l_2, \cdots, l_{n-1}) \in \mathbb{R}^{n-1}$ is NE iff x_k satisfy the following inequalities for all $k \in \{1, 2, \cdots, n-1\}$

$$x_k \geq -\alpha_{kk}^{d_1}(m_k - 1) - \sum_{j \in N_1, j \neq k} \alpha_{kj}^{d_1} m_j + \sum_{j \in N_2} \alpha_{kj}^{d_2} m_j + \epsilon_k(d_2) - \epsilon_k(d_1), \;\; if \;\; k \in N_1,$$

[2]See Appendix B to clarify more the case of two types and three types of players.

and

$$x_k \leq -\sum_{j\in N_1} \alpha_{kj}^{d_1} m_j + \sum_{j\in N_2, j\neq k} \alpha_{kj}^{d_2} m_j + \alpha_{kk}^{d_2}(m_k - 1) + \epsilon_k(d_2) - \epsilon_k(d_1), \ \ if \ \ k \in N_2.$$

Now, we need to show it is true for all n, so we have two main cases:

Case 1: If players of type t_k make decision d_1, i.e. $k \in N_1$, then

$$\Pi_k(d_1; l_1, l_2, \cdots, m_k, \cdots, l_{n-1}, l_n) \geq \Pi_k(d_2; l_1, l_2, \cdots, m_k - 1, \cdots, l_{n-1}, l_n).$$

Substituting the utilities (3.5) and (3.6) in the previous inequalities, we get

$$\begin{aligned}
&\omega_k^{d_1} + \alpha_{k1}^{d_1} l_1 + \alpha_{k2}^{d_1} l_2 + \cdots + \alpha_{kk}^{d_1}(m_k - 1) + \cdots + \alpha_{kn}^{d_1} l_n \geq \\
&\omega_k^{d_2} + \alpha_{k1}^{d_2}(m_1 - l_1) + \alpha_{k2}^{d_2}(m_2 - l_2) + \cdots + \alpha_{kk}^{d_2}(m_k - (m_k - 1) - 1) + \cdots \\
&\quad + \alpha_{kn}^{d_2}(m_n - l_n) + \epsilon_k(d_2) - \epsilon_k(d_1),
\end{aligned}$$

which in other way can be written as

$$\begin{aligned}
&\Pi_k(d_1; l_1, l_2, \cdots, m_k, \cdots, l_{n-1}) + \alpha_{kn}^{d_1} l_n \geq \\
&\Pi_k(d_2; l_1, l_2, \cdots, m_k - 1, \cdots, l_{n-1}) + \alpha_{kn}^{d_2}(m_n - l_n).
\end{aligned}$$

Using Definition 3.5 for x_k we get the following inequality

$$\begin{aligned}
x_k \geq &- \alpha_{kk}^{d_1}(m_k - 1) - \sum_{j\in N_1, j\neq k} \alpha_{kj}^{d_1} m_j + \sum_{j\in N_2} \alpha_{kj}^{d_2} m_j + \alpha_{kn}^{d_2}(m_n - l_n) - \alpha_{kn}^{d_1} l_n \\
&+ \epsilon_k(d_2) - \epsilon_k(d_1).
\end{aligned}$$

Now, there are two subcases:

Case 1-1: If players of type t_n make decision d_1 ($n \in N_1$), i.e. $l_n = m_n$, then

$$
\begin{aligned}
x_k &\geq -\alpha_{kk}^{d_1}(m_k-1) - \sum_{j\in N_1, j\neq k} \alpha_{kj}^{d_1} m_j + \sum_{j\in N_2} \alpha_{kj}^{d_2} m_j + \alpha_{kn}^{d_2}(m_n - m_n) \\
&\quad -\alpha_{kn}^{d_1} m_n + \epsilon_k(d_2) - \epsilon_k(d_1) \\
\Longrightarrow x_k &\geq -\alpha_{kk}^{d_1}(m_k-1) - \sum_{j\in N_1, j\neq k} \alpha_{kj}^{d_1} m_j + \sum_{j\in N_2} \alpha_{kj}^{d_2} m_j + \alpha_{kn}^{d_2}(0) - \alpha_{kn}^{d_1} m_n \\
&\quad +\epsilon_k(d_2) - \epsilon_k(d_1) \\
\Longrightarrow x_k &\geq -\alpha_{kk}^{d_1}(m_k-1) - \sum_{j\in N_1, j\neq k} \alpha_{kj}^{d_1} m_j + \sum_{j\in N_2} \alpha_{kj}^{d_2} m_j \\
&\quad +\epsilon_k(d_2) - \epsilon_k(d_1) \quad \text{where } n \in N_1
\end{aligned}
$$

<u>Case 1-2</u>: If players of type t_n make decision d_2 ($n \in N_2$), i.e. $l_n = 0$, then

$$
\begin{aligned}
x_k &\geq -\alpha_{kk}^{d_1}(m_k-1) - \sum_{j\in N_1, j\neq k} \alpha_{kj}^{d_1} m_j + \sum_{j\in N_2} \alpha_{kj}^{d_2} m_j + \alpha_{kn}^{d_2}(m_n - 0) \\
&\quad -\alpha_{kn}^{d_1} 0 + \epsilon_k(d_2) - \epsilon_k(d_1) \\
\Longrightarrow x_k &\geq -\alpha_{kk}^{d_1}(m_k-1) - \sum_{j\in N_1, j\neq k} \alpha_{kj}^{d_1} m_j + \sum_{j\in N_2} \alpha_{kj}^{d_2} m_j + \alpha_{kn}^{d_2}(m_n) - 0 \\
&\quad +\epsilon_k(d_2) - \epsilon_k(d_1) \\
\Longrightarrow x_k &\geq -\alpha_{kk}^{d_1}(m_k-1) - \sum_{j\in N_1, j\neq k} \alpha_{kj}^{d_1} m_j + \sum_{j\in N_2} \alpha_{kj}^{d_2} m_j \\
&\quad +\epsilon_k(d_2) - \epsilon_k(d_1) \quad \text{where } n \in N_2
\end{aligned}
$$

<u>Case 2</u>: If players of type t_k make decision d_2, i.e. $k \in N_2$, then

$$
\Pi_k(d_1; l_1, l_2, \cdots, l_k + 1, \cdots, l_{n-1}, l_n) \leq \Pi_k(d_2; l_1, l_2, \cdots, l_k, \cdots, l_{n-1}, l_n)
$$

Substituting the utilities (3.5) and (3.6) in the previous inequality, we get

$$
\begin{aligned}
&\omega_k^{d_1} + \alpha_{k1}^{d_1} l_1 + \alpha_{k2}^{d_1} l_2 + \cdots + \alpha_{kk}^{d_1}(l_k + 1 - 1) + \cdots + \alpha_{kn}^{d_1} l_n \leq \\
&\omega_k^{d_2} + \alpha_{k1}^{d_2}(m_1 - l_1) + \alpha_{k2}^{d_2}(m_2 - l_2) + \cdots + \alpha_{kk}^{d_2}(m_k - l_k - 1) + \cdots \\
&+ \alpha_{kn}^{d_2}(m_n - l_n) + \epsilon_k(d_2) - \epsilon_k(d_1),
\end{aligned}
$$

which in other way can be written as

$$\begin{aligned}&\Pi_k(d_1; l_1, l_2, \cdots, l_k+1, \cdots, l_{n-1}) + \alpha_{kn}^{d_1} l_n \leq \\ &\Pi_k(d_2; l_1, l_2, \cdots, l_k, \cdots, l_{n-1}) + \alpha_{kn}^{d_2}(m_n - l_n).\end{aligned}$$

Using the Definition 3.5 of x_k we get the following inequality

$$x_k \leq \alpha_{kk}^{d_2}(m_k-1) - \sum_{j\in N_1} \alpha_{kj}^{d_1} m_j + \sum_{j\in N_2, j\neq k} \alpha_{kj}^{d_2} m_j - \alpha_{kn}^{d_1} l_n + \alpha_{kn}^{d_2}(m_n - l_n) + \epsilon_k(d_2) - \epsilon_k(d_1).$$

Now, there are two subcases:

Case 2-1: If players of type t_n make decision d_1 ($n \in N_1$), i.e. $l_n = m_n$, then

$$\begin{aligned}
x_k &\leq \alpha_{kk}^{d_2}(m_k-1) - \sum_{j\in N_1} \alpha_{kj}^{d_1} m_j + \sum_{j\in N_2, j\neq k} \alpha_{kj}^{d_2} m_j - \alpha_{kn}^{d_1} m_n \\
&\quad + \alpha_{kn}^{d_2}(m_n - m_n) + \epsilon_k(d_2) - \epsilon_k(d_1) \\
\Longrightarrow x_k &\leq \alpha_{kk}^{d_2}(m_k-1) - \sum_{j\in N_1} \alpha_{kj}^{d_1} m_j + \sum_{j\in N_2, j\neq k} \alpha_{kj}^{d_2} m_j - \alpha_{kn}^{d_1} m_n + 0 \\
&\quad + \epsilon_k(d_2) - \epsilon_k(d_1) \\
\Longrightarrow x_k &\leq \alpha_{kk}^{d_2}(m_k-1) - \sum_{j\in N_1} \alpha_{kj}^{d_1} m_j + \sum_{j\in N_2, j\neq k} \alpha_{kj}^{d_2} m_j \\
&\quad + \epsilon_k(d_2) - \epsilon_k(d_1) \quad \text{where } n \in N_1
\end{aligned}$$

Case 2-2: If players of type t_n make decision d_2 ($n \in N_2$), i.e. $l_n = 0$, then

$$\begin{aligned}
x_k &\leq \alpha_{kk}^{d_2}(m_k-1) - \sum_{j\in N_1} \alpha_{kj}^{d_1} m_j + \sum_{j\in N_2, j\neq k} \alpha_{kj}^{d_2} m_j - \alpha_{kn}^{d_1} 0 \\
&\quad + \alpha_{kn}^{d_2}(m_n - 0) + \epsilon_k(d_2) - \epsilon_k(d_1) \\
\Longrightarrow x_k &\leq \alpha_{kk}^{d_2}(m_k-1) - \sum_{j\in N_1} \alpha_{kj}^{d_1} m_j + \sum_{j\in N_2, j\neq k} \alpha_{kj}^{d_2} m_j - 0 + \alpha_{kn}^{d_2} m_n \\
&\quad + \epsilon_k(d_2) - \epsilon_k(d_1) \\
\Longrightarrow x_k &\leq \alpha_{kk}^{d_2}(m_k-1) - \sum_{j\in N_1} \alpha_{kj}^{d_1} m_j + \sum_{j\in N_2, j\neq k} \alpha_{kj}^{d_2} m_j \\
&\quad + \epsilon_k(d_2) - \epsilon_k(d_1) \quad \text{where } n \in N_2
\end{aligned}$$

combining the two cases, we conclude that the united stratgy $(l_1, l_2, \cdots, l_n) \in \mathbb{R}^n$ is NE iff

$$X_k(l_1, \cdots, l_{k-1}, m_k, l_{k+1}, \cdots, l_n) = -\alpha_{kk}^{d_1}(m_k - 1) - \sum_{j \in N_1, j \neq k} \alpha_{kj}^{d_1} m_j + \sum_{j \in N_2} \alpha_{kj}^{d_2} m_j$$

$$+\epsilon_k(d_2) - \epsilon_k(d_1), \qquad if \;\; k \in N_1,$$

and

$$X_k(l_1, \cdots, l_{k-1}, 0, l_{k+1} \cdots, l_n) = -\sum_{j \in N_1} \alpha_{kj}^{d_1} m_j + \sum_{j \in N_2, j \neq k} \alpha_{kj}^{d_2} m_j + \alpha_{kk}^{d_2}(m_k - 1)$$

$$+\epsilon_k(d_2) - \epsilon_k(d_1), \qquad if \;\; k \in N_2,$$

which completes the proof. □

Recall that the Nash region $\boldsymbol{NR}(S)$ of a strategy $S \in \boldsymbol{S}$ is the set of all preferences $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ for which S is Nash Equilibrium.
We will represent special case of united strategies for n−types of players in the following result.

Lemma 3.13. *A united strategy $(m_1, m_2, \cdots, \cdots, m_n)$ is Nash Equilibrium iff $x \in \boldsymbol{NR}(m_1, m_2, \cdots, m_n)$, where the Nash region $\boldsymbol{NR}(m_1, m_2, \cdots, m_n)$ is given by*

$$\begin{aligned} \boldsymbol{NR}(m_1, m_2, \cdots, m_n) = \{x \in \mathbb{R}^n : x_1 &\geq X_1(m_1, m_2, \cdots, m_n) \;\; and \\ x_2 &\geq X_2(m_1, m_2, \cdots, m_n) \;\; and \\ &\vdots \\ x_n &\geq X_n(m_1, m_2, \cdots, m_n)\}. \end{aligned}$$

Proof. Consider the following united strategy: $S_U(m_1, m_2, \cdots, m_n) \in \mathbb{R}^n$ strategy, where all players of all types make the decision d_1. The united strategy $S_U =$

$(m_1, m_2, \cdots, m_n)$ is NE iff

$$\begin{aligned}
\Pi_1(d_1; m_1, m_2, \cdots, m_n) &\geq \Pi_1(d_2; m_1-1, m_2, \cdots, m_n) \;\; and \\
\Pi_2(d_1; m_1, m_2, \cdots, m_n) &\geq \Pi_2(d_2; m_1, m_2-1, \cdots, m_n) \;\; and \\
&\vdots \\
\Pi_n(d_1; m_1, m_2, \cdots, m_n) &\geq \Pi_n(d_2; m_1, m_2, \cdots, m_n-1).
\end{aligned} \tag{3.13}$$

Substituting the utility functions given in (3.5) and (3.6) in inequality (3.13) we get

$$\begin{aligned}
&\omega_1^{d_1} + \alpha_{11}^{d_1}(m_1-1) + \alpha_{12}^{d_1}m_2 + \cdots + \alpha_{1n}^{d_1}m_n + \epsilon_1(d_1) \geq \\
&\omega_1^{d_2} + \alpha_{11}^{d_2}(m_1-(m_1-1)-1) + \alpha_{12}^{d_2}(m_2-m_2) + \cdots + \alpha_{1n}^{d_2}(m_n-m_n) + \epsilon_1(d_2) \; and \\
&\omega_2^{d_1} + \alpha_{21}^{d_1}m_1 + \alpha_{22}^{d_1}(m_2-1) + \cdots + \alpha_{2n}^{d_1}m_n + \epsilon_2(d_1) \geq \\
&\omega_2^{d_2} + \alpha_{21}^{d_2}(m_1-m_1) + \alpha_{22}^{d_2}(m_2-(m_2-1)-1) + \cdots + \alpha_{2n}^{d_2}(m_n-m_n) + \epsilon_2(d_2) \; and \\
&\vdots \\
&\omega_n^{d_1} + \alpha_{n1}^{d_1}m_1 + \alpha_{n2}^{d_1}m_2 + \cdots + \alpha_{nn}^{d_1}(m_n-1) + \epsilon_n(d_1) \geq \\
&\omega_n^{d_2} + \alpha_{n1}^{d_2}(m_1-m_1) + \alpha_{n2}^{d_2}(m_2-m_2) + \cdots + \alpha_{nn}^{d_2}(m_n-(m_n-1)-1) + \epsilon_n(d_2).
\end{aligned} \tag{3.14}$$

Rearrange the previous inequalities (3.14) we obtain

$$\begin{aligned}
\omega_1^{d_1} - \omega_1^{d_2} &\geq -\alpha_{11}^{d_1}(m_1-1) - \alpha_{12}^{d_1}m_2 - \cdots - \alpha_{1n}^{d_1}m_n + \epsilon_1(d_2) - \epsilon_1(d_1) \\
\omega_2^{d_1} - \omega_2^{d_2} &\geq -\alpha_{21}^{d_1}m_1 - \alpha_{22}^{d_1}(m_2-1) - \cdots - \alpha_{2n}^{d_1}m_n + \epsilon_2(d_2) - \epsilon_2(d_1) \\
&\vdots \\
\omega_n^{d_1} - \omega_n^{d_2} &\geq -\alpha_{n1}^{d_1}m_1 - \alpha_{n2}^{d_1}m_2 - \cdots - \alpha_{nn}^{d_1}(m_n-1) + \epsilon_n(d_2) - \epsilon_n(d_1).
\end{aligned}$$

Substituting the values of the relative decisions from Definition 3.5, the last inequalities simplifying to

$$\begin{aligned} x_1 &\geq X_1(m_1, m_2, \cdots, m_n) \quad \textit{and} \\ x_2 &\geq X_2(m_1, m_2, \cdots, m_n) \quad \textit{and} \\ &\vdots \\ x_n &\geq X_n(m_1, m_2, \cdots, m_n), \end{aligned}$$

where the strategic thresholds of the $(m_1, m_2, \cdots, m_n)$ strategy are given, respectively, by

$$\begin{aligned} X_1(m_1, m_2, \cdots, m_n) &= -\alpha_{11}^{d_1}(m_1 - 1) - \alpha_{12}^{d_1} m_2 - \cdots - \alpha_{1n}^{d_1} m_n + \epsilon_1(d_2) - \epsilon_1(d_1) \\ X_2(m_1, m_2, \cdots, m_n) &= -\alpha_{21}^{d_1} m_1 - \alpha_{22}^{d_1}(m_2 - 1) - \cdots - \alpha_{2n}^{d_1} m_n + \epsilon_2(d_2) - \epsilon_2(d_1) \\ &\vdots \\ X_n(m_1, m_2, \cdots, m_n) &= -\alpha_{n1}^{d_1} m_1 - \alpha_{n2}^{d_1} m_2 - \cdots - \alpha_{nn}^{d_1}(m_n - 1) + \epsilon_n(d_2) - \epsilon_n(d_1). \end{aligned}$$

Hence, the corresponding Nash region is

$$\begin{aligned} \boldsymbol{NR}(m_1, m_2, \cdots, m_n) = \{x \in \mathbb{R}^n : x_1 &\geq X_1(m_1, m_2, \cdots, m_n) \quad \textit{and} \\ x_2 &\geq X_2(m_1, m_2, \cdots, m_n) \quad \textit{and} \\ &\vdots \\ x_n &\geq X_n(m_1, m_2, \cdots, m_n)\}. \end{aligned}$$

□

Note that the Lemma 3.13 characterizes the conditions where all players of all types agree to make one decision which is d_1. In the following result we will study the opposite extreme case.

Lemma 3.14. *A united strategy* $S_U = (0, 0, \cdots, 0) \in \mathbb{R}^n$ *is Nash Equilibrium iff* $x \in \boldsymbol{NR}(0, 0, \cdots, 0)$, *where the Nash region* $\boldsymbol{NR}(0, 0, \cdots, 0)$ *is given by*

$$\begin{aligned}\boldsymbol{NR}(0,0,\cdots,0) = \{x \in \mathbb{R}^n : x_1 &\leq X_1(0,0,\cdots,0) \quad \textit{and} \\ x_2 &\leq X_2(0,0,\cdots,0) \quad \textit{and} \\ &\vdots \\ x_n &\leq X_n(0,0,\cdots,0)\}.\end{aligned}$$

Proof. Consider the following united strategy: $S_U = (0, 0, \cdots, 0) \in \mathbb{R}^n$ strategy, where all players of all types make the decision d_2. The united strategy $S_U = (0, 0, \cdots, 0) \in \mathbb{R}^n$ is NE iff

$$\begin{aligned}\Pi_1(d_2; 0,0,\cdots,0) &\geq \Pi_1(d_1; 1,0,\cdots,0) \quad \textit{and} \\ \Pi_2(d_2; 0,0,\cdots,0) &\geq \Pi_2(d_1; 0,1,\cdots,0) \quad \textit{and} \\ &\vdots \\ \Pi_n(d_2; 0,0,\cdots,0) &\geq \Pi_n(d_1; 0,0,\cdots,1).\end{aligned} \tag{3.15}$$

Substituting the utility functions given in (3.5) and (3.6) in inequality (3.15) we obtain

$$\begin{aligned}&\omega_1^{d_2} + \alpha_{11}^{d_2}(m_1 - 1) + \alpha_{12}^{d_2}(m_2 - 0) + \cdots + \alpha_{1n}^{d_2}(m_n - 0) + \epsilon_1(d_2) \geq \\ &\omega_1^{d_1} + \alpha_{11}^{d_1}(1 - 1) + \alpha_{12}^{d_1}(0) + \cdots + \alpha_{1n}^{d_1}(0) + \epsilon_1(d_1) \ \textit{and} \\ &\omega_2^{d_2} + \alpha_{21}^{d_2}(m_1 - 0) + \alpha_{22}^{d_2}(m_2 - 1) + \cdots + \alpha_{2n}^{d_2}(m_n - 0) + \epsilon_2(d_2) \geq \\ &\omega_2^{d_1} + \alpha_{21}^{d_1}(0) + \alpha_{12}^{d_1}(1 - 1) + \cdots + \alpha_{2n}^{d_1}(0) + \epsilon_2(d_1) \ \textit{and} \\ &\vdots \\ &\omega_n^{d_2} + \alpha_{n1}^{d_2}(m_1 - 0) + \alpha_{n2}^{d_2}(m_2 - 0) + \cdots + \alpha_{nn}^{d_2}(m_n - 1) + \epsilon_n(d_2) \geq \\ &\omega_n^{d_1} + \alpha_{n1}^{d_1}(0) + \alpha_{n2}^{d_1}(0) + \cdots + \alpha_{nn}^{d_1}(1 - 1) + \epsilon_n(d_1).\end{aligned} \tag{3.16}$$

Rearrange the previous inequalities (3.16) we get

$$\begin{aligned}
\omega_1^{d_1} - \omega_1^{d_2} &\leq \alpha_{11}^{d_2}(m_1 - 1) + \alpha_{12}^{d_2} m_2 + \cdots + \alpha_{1n}^{d_2} m_n + \epsilon_1(d_2) - \epsilon_1(d_1) \ \ \textit{and} \\
\omega_2^{d_1} - \omega_2^{d_2} &\leq \alpha_{21}^{d_2} m_1 + \alpha_{22}^{d_2}(m_2 - 1) + \cdots + \alpha_{2n}^{d_2} m_n + \epsilon_2(d_2) - \epsilon_2(d_1) \ \ \textit{and} \\
&\vdots \\
\omega_n^{d_1} - \omega_n^{d_2} &\leq \alpha_{n1}^{d_2} m_1 + \alpha_{n2}^{d_2} m_2 + \cdots + \alpha_{nn}^{d_2}(m_n - 1) + \epsilon_n(d_2) - \epsilon_n(d_1).
\end{aligned}$$

Substituting the values of the relative decisions from Definition 3.5, the last inequalities simplifying to

$$\begin{aligned}
x_1 &\leq X_1(0, 0, \cdots, 0) \ \ \textit{and} \\
x_2 &\leq X_2(0, 0, \cdots, 0) \ \ \textit{and} \\
&\vdots \\
x_n &\leq X_n(0, 0, \cdots, 0),
\end{aligned}$$

where the strategic thresholds of the $S_U = (0, 0, \cdots, 0)$ strategy are given, respectively, by

$$\begin{aligned}
X_1(0, 0, \cdots, 0) &= \alpha_{11}^{d_2}(m_1 - 1) + \alpha_{12}^{d_2} m_2 + \cdots + \alpha_{1n}^{d_2} m_n + \epsilon_1(d_2) - \epsilon_1(d_1) \ \ \textit{and} \\
X_2(0, 0, \cdots, 0) &= \alpha_{21}^{d_2} m_1 + \alpha_{22}^{d_2}(m_2 - 1) + \cdots + \alpha_{2n}^{d_2} m_n + \epsilon_2(d_2) - \epsilon_2(d_1) \ \ \textit{and} \\
&\vdots \\
X_n(0, 0, \cdots, 0) &= \alpha_{n1}^{d_2} m_1 + \alpha_{n2}^{d_2} m_2 + \cdots + \alpha_{nn}^{d_2}(m_n - 1) + \epsilon_n(d_2) - \epsilon_n(d_1).
\end{aligned}$$

Hence, the corresponding Nash region is

$$\begin{aligned}
\boldsymbol{NR}(0, 0, \cdots, 0) = \{x \in \mathbb{R}^n : x_1 &\leq X_1(0, 0, \cdots, 0) \ \ \textit{and} \\
x_2 &\leq X_2(0, 0, \cdots, 0) \ \ \textit{and} \\
&\vdots \\
x_n &\leq X_n(0, 0, \cdots, 0)\}.
\end{aligned}$$

□

For more cases, see the appendix C.

3.3 Example for three types of players

In this section we will take a special case for three types of players. We will construct Nash regions $\boldsymbol{NR}(S_U)$ for the united strategy $S_U \in \overline{\boldsymbol{S}_U^n}$ and characterize such Nash regions geometrically. The united Nash regions are shown in Figure 3.1 using the laten numerization for eight octants:

We observe that there are eight (2^3) united strategies listed as the follows

1. I for 1^{st} octant represents (d_1,d_1,d_1) strategy - all players choose decision d_1;
2. II for 2^{nd} octant represents (d_2,d_1,d_1) strategy - all players with type t_2 and t_3 choose decision d_1 but all players with type t_1 choose decision d_2;
3. III for 3^{rd} octant represents (d_2,d_2,d_1) strategy - all players with type t_3 choose decision d_1 but all players with type t_1 and t_2 choose decision d_2;

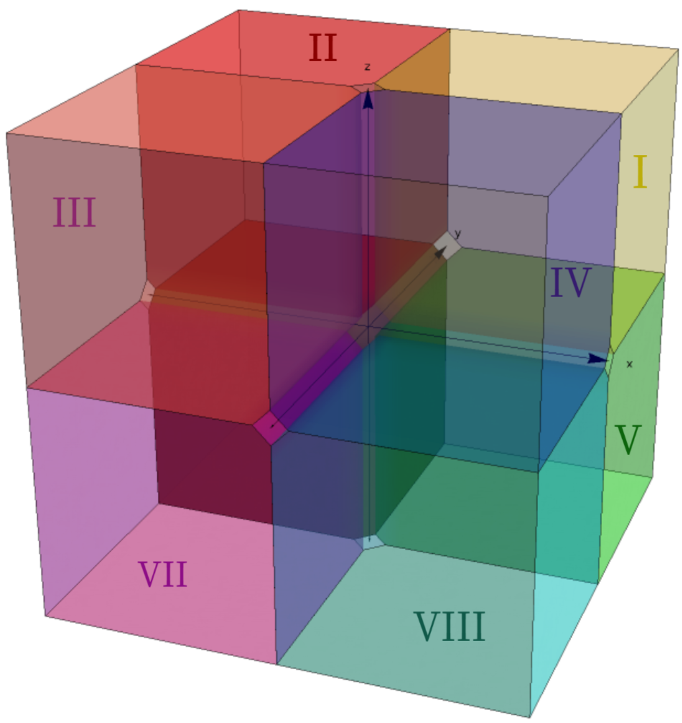


FIGURE 3.1: United Nash regions in three dimensions for three types of homogenous players.

4. IV for 4^{th} octant represents (d_1,d_2,d_1) strategy - all players with type t_1 and t_3 choose decision d_1 but all players with type t_2 choose decision d_2;

5. V for 5^{th} octant represents (d_1,d_1,d_2) strategy - all players with type t_1 and t_2 choose decision d_1 but all players with type t_3 choose decision d_2;

6. VI for 6^{th} octant represents (d_2,d_1,d_2) strategy - all players with type t_2 choose decision d_1 but all players with type t_1 and t_3 choose decision d_2;

7. VII for 7^{th} octant represents (d_2,d_2,d_2) strategy - all players choose decision d_2;

8. VIII for 8^{th} octant represents (d_1,d_2,d_2) strategy - all players with type t_1 choose decision d_1 but all players with type t_2 and t_3 choose decision d_2.

The united strategy (m_1, m_2, m_3) implies that all individuals of three types make the decision d_1, so we may refer to this strategy by (d_1, d_1, d_1). Also, for simplicity to follow our model we denote to the corresponding horizontal threshold by $X(d_1, d_1, d_1)$, vertical threshold by $Y(d_1, d_1, d_1)$ and depth threshold by $Z(d_1, d_1, d_1)$. Using Theorem 3.12, these thresholds can be writing as

$$\begin{aligned}
X(d_1, d_1, d_1) &= -\alpha_{11}^{d_1}(m_1 - 1) - \alpha_{12}^{d_1} m_2 - \alpha_{13}^{d_1} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1), \\
Y(d_1, d_1, d_1) &= -\alpha_{21}^{d_1} m_1 - \alpha_{22}^{d_1}(m_2 - 1) - \alpha_{23}^{d_1} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1), \\
Z(d_1, d_1, d_1) &= -\alpha_{31}^{d_1} m_1 - \alpha_{32}^{d_1} m_2 - \alpha_{33}^{d_1}(m_3 - 1) + \epsilon_3(d_2) - \epsilon_3(d_1).
\end{aligned}$$

Furthermore, the Nash region $\boldsymbol{NR}(d_1, d_1, d_1)$ includes the first octant I in Figure 3.1 and is given by

$$\begin{aligned}
\boldsymbol{NR}(d_1, d_1, d_1) = \{(x, y, z) \in \mathbb{R}^3 \;\; : \;\; & x \geq X(d_1, d_1, d_1), \\
& y \geq Y(d_1, d_1, d_1), \\
& z \geq Z(d_1, d_1, d_1)\}.
\end{aligned}$$

Hence, the united strategy (d_1, d_1, d_1) is a Nash equilibrium if and only if $(x, y, z) \in \boldsymbol{NR}(d_1, d_1, d_1)$ which is illustrated in Figure 3.2. The code used to generate this

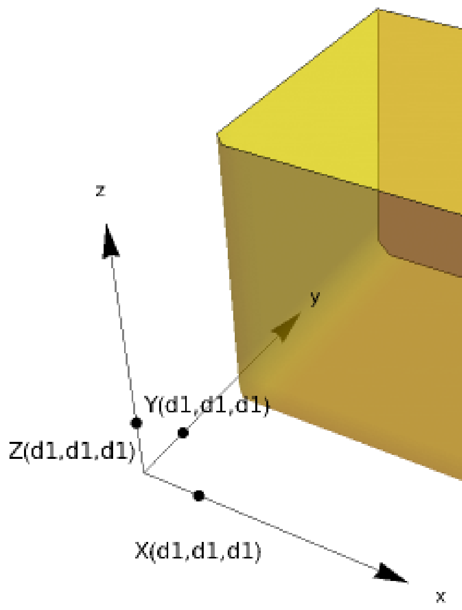


FIGURE 3.2: The united Nash equilibria region $\boldsymbol{NR}(d_1, d_1, d_1)$ in space.

figure in Mathematica is shown in Appendix D.

The united strategy $(0, m_2, m_3)$ implies that all players of type t_1 make decision d_2, but all players of types t_2 and t_3 make the decision d_1, so we may refer to this strategy by (d_2, d_1, d_1). Also, for simplicity to follow our model we denote to the corresponding horizontal threshold by $X(d_2, d_1, d_1)$, vertical threshold by $Y(d_2, d_1, d_1)$ and depth threshold by $Z(d_2, d_1, d_1)$. Using Theorem 3.12, these thresholds can be writing as

$$\begin{aligned} X(d_2, d_1, d_1) &= \alpha_{11}^{d_2}(m_1 - 1) - \alpha_{12}^{d_1} m_2 - \alpha_{13}^{d_1} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1), \\ Y(d_2, d_1, d_1) &= \alpha_{21}^{d_2} m_1 - \alpha_{22}^{d_1}(m_2 - 1) - \alpha_{23}^{d_1} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1), \\ Z(d_2, d_1, d_1) &= \alpha_{31}^{d_2} m_1 - \alpha_{32}^{d_1} m_2 - \alpha_{33}^{d_1}(m_3 - 1) + \epsilon_3(d_2) - \epsilon_3(d_1). \end{aligned}$$

Furthermore, the Nash region $\boldsymbol{NR}(d_2, d_1, d_1)$ is the second Octant II in Figure 3.1 and is given by

$$\begin{aligned} \boldsymbol{NR}(d_2, d_1, d_1) = \{(x, y, z) \in \mathbb{R}^3 \ : \ & x \le X(d_2, d_1, d_1), \\ & y \ge Y(d_2, d_1, d_1), \\ & z \ge Z(d_2, d_1, d_1)\}. \end{aligned}$$

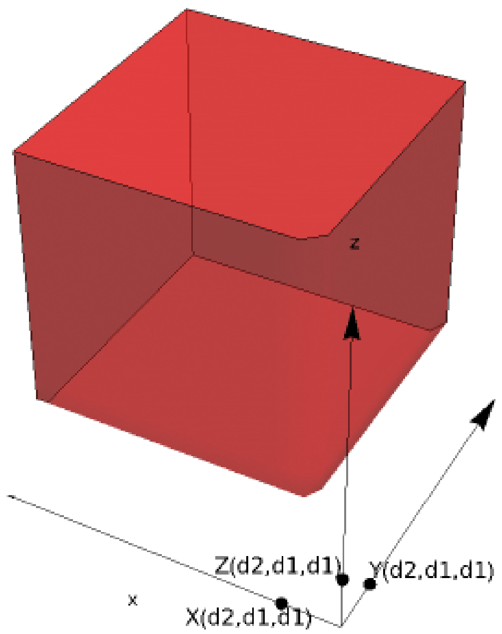


FIGURE 3.3: The united Nash equilibria region $\boldsymbol{NR}(d_2, d_1, d_1)$ in space.

Hence, the united strategy (d_2, d_1, d_1) is a Nash equilibrium if and only if $(x, y, z) \in \boldsymbol{NR}(d_2, d_1, d_1)$ which is illustrated in Figure 3.3. The code used to generate this figure in Mathematica is shown in Appendix D.

The united strategy $(0, 0, m_3)$ which implies all players of type t_1 and t_2 make decision d_2, but all players of types t_3 make the decision d_1, so we may refer to this strategy by (d_2, d_2, d_1). Also, for simplicity to follow our model we denote to the corresponding horizontal threshold by $X(d_2, d_2, d_1)$, vertical threshold by $Y(d_2, d_2, d_1)$ and depth threshold by $Z(d_2, d_2, d_1)$. Using Theorem 3.12, these thresholds can be writing as

$$\begin{aligned}
X(d_2, d_2, d_1) &= \alpha_{11}^{d_2}(m_1 - 1) + \alpha_{12}^{d_2} m_2 - \alpha_{13}^{d_1} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1), \\
Y(d_2, d_2, d_1) &= \alpha_{21}^{d_2} m_1 + \alpha_{22}^{d_2}(m_2 - 1) - \alpha_{23}^{d_1} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1), \\
Z(d_2, d_2, d_1) &= \alpha_{31}^{d_2} m_1 + \alpha_{32}^{d_2} m_2 - \alpha_{33}^{d_1}(m_3 - 1) + \epsilon_3(d_2) - \epsilon_3(d_1).
\end{aligned}$$

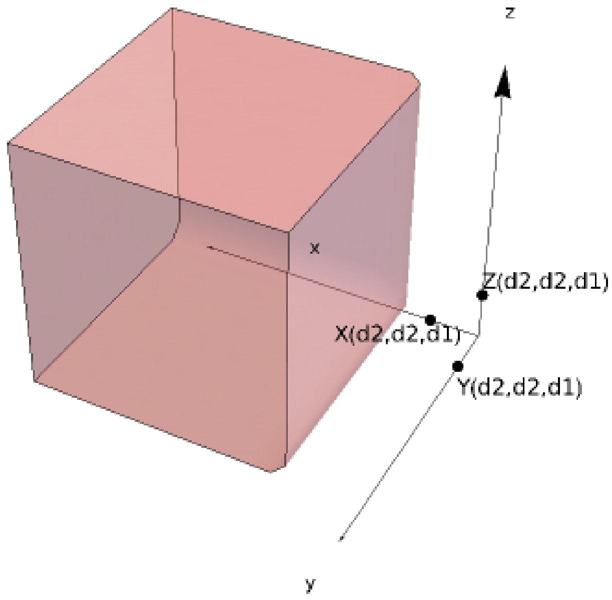


FIGURE 3.4: The united Nash equilibria region $\boldsymbol{NR}(d_2, d_2, d_1)$ in space.

Furthermore, the Nash region $\boldsymbol{NR}(d_2, d_2, d_1)$ includes the third octant III in Figure 3.1 and is given by

$$\begin{aligned}\boldsymbol{NR}(d_2, d_2, d_1) = \{(x, y, z) \in \mathbb{R}^3 \quad : \quad & x \leq X(d_2, d_2, d_1), \\ & y \leq Y(d_2, d_2, d_1), \\ & z \geq Z(d_2, d_2, d_1)\}.\end{aligned}$$

Hence, the united strategy (d_2, d_2, d_1) is a Nash equilibrium if and only if $(x, y, z) \in \boldsymbol{NR}(d_2, d_2, d_1)$ which is illustrated in Figure 3.4. The code used to generate this figure in Mathematica is shown in Appendix D.

The united strategy $(m_1, 0, m_3)$ implies that all players of type t_1 and t_3 make decision d_1, but all players of types t_2 make the decision d_2, so we may refer to this strategy by (d_1, d_2, d_1). Also, for simplicity to follow our model we denote to the corresponding horizontal threshold by $X(d_1, d_2, d_1)$, vertical threshold by $Y(d_1, d_2, d_1)$ and depth threshold by $Z(d_1, d_2, d_1)$. Using Theorem 3.12, these

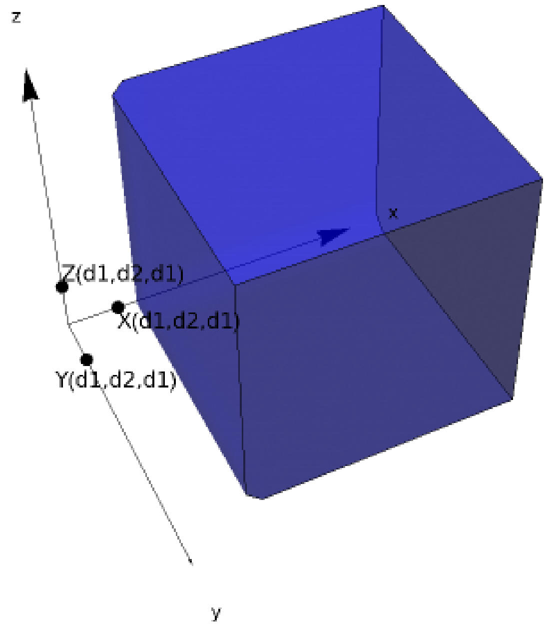


FIGURE 3.5: The united Nash equilibria region $\boldsymbol{NR}(d_1, d_2, d_1)$ in space.

thresholds can be writing as

$$\begin{aligned} X(d_1, d_2, d_1) &= -\alpha_{11}^{d_1}(m_1 - 1) + \alpha_{12}^{d_2} m_2 - \alpha_{13}^{d_1} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1), \\ Y(d_1, d_2, d_1) &= -\alpha_{21}^{d_1} m_1 + \alpha_{22}^{d_2}(m_2 - 1) - \alpha_{23}^{d_1} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1), \\ Z(d_1, d_2, d_1) &= -\alpha_{31}^{d_1} m_1 + \alpha_{32}^{d_2} m_2 - \alpha_{33}^{d_1}(m_3 - 1) + \epsilon_3(d_2) - \epsilon_3(d_1). \end{aligned}$$

Furthermore, the Nash region $\boldsymbol{NR}(d_1, d_2, d_1)$ includes the fourth octant IV in Figure 3.1 and is given by

$$\begin{aligned} \boldsymbol{NR}(d_1, d_2, d_1) = \{(x, y, z) \in \mathbb{R}^3 \quad : \quad & x \geq X(d_1, d_2, d_1), \\ & y \leq Y(d_1, d_2, d_1), \\ & z \geq Z(d_1, d_2, d_1)\}. \end{aligned}$$

Hence, the united strategy $\boldsymbol{NR}(d_1, d_2, d_1)$ is a Nash equilibrium if and only if $(x, y, z) \in \boldsymbol{NR}(d_1, d_2, d_1)$ which is illustrated in Figure 3.5. The code used to generate this figure in Mathematica is shown in Appendix D.

The united strategy $(m_1, m_2, 0)$ implies that all players of type t_1 and t_2 makes the decision d_1, but all players of type t_3 make the decision d_2, so we may refer to this strategy by (d_1, d_1, d_2). Also, for simplicity to follow our model we denote to the Horizontal corresponding threshold by $X(d_1, d_1, d_2)$, vertical threshold by $Y(d_1, d_1, d_2)$ and depth threshold by $Z(d_1, d_1, d_2)$. Using Theorem 3.12, these thresholds can be writing as

$$\begin{aligned} X(d_1, d_1, d_2) &= -\alpha_{11}^{d_1}(m_1 - 1) - \alpha_{12}^{d_1} m_2 + \alpha_{13}^{d_2} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1), \\ Y(d_1, d_1, d_2) &= -\alpha_{21}^{d_1} m_1 - \alpha_{22}^{d_1}(m_2 - 1) + \alpha_{23}^{d_2} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1), \\ Z(d_1, d_1, d_2) &= -\alpha_{31}^{d_1} m_1 - \alpha_{32}^{d_1} m_2 + \alpha_{33}^{d_2}(m_3 - 1) + \epsilon_3(d_2) - \epsilon_3(d_1). \end{aligned}$$

Furthermore, the Nash region $\boldsymbol{NR}(d_1, d_1, d_2)$ includes the fifth octant V in Figure 3.1 and is given by

$$\begin{aligned} \boldsymbol{NR}(d_1, d_1, d_2) = \{(x, y, z) \in \mathbb{R}^3 \quad : \quad & x \geq X(d_1, d_1, d_2), \\ & y \geq Y(d_1, d_1, d_2), \\ & z \leq Z(d_1, d_1, d_2)\}. \end{aligned}$$

Hence, the united strategy (d_1, d_1, d_2) is a Nash equilibrium if and only if $(x, y, z) \in \boldsymbol{NR}(d_1, d_1, d_2)$ which is illustrated in Figure 3.6. The code used to generate this figure in Mathematica is shown in Appendix D.

The united strategy $(0, m_2, 0)$ implies that all players of types t_1 and t_3 make decision d_2, but all players of type t_2 make the decision d_1, so we may refer to this strategy by (d_2, d_1, d_2). Also, for simplicity to follow our model we denote to the corresponding horizontal threshold by $X(d_2, d_1, d_2)$, vertical threshold by $Y(d_2, d_1, d_2)$ and depth threshold by $Z(d_2, d_1, d_2)$. Using Theorem 3.12, these thresholds can be writing as

$$\begin{aligned} X(d_2, d_1, d_2) &= \alpha_{11}^{d_2}(m_1 - 1) - \alpha_{12}^{d_1} m_2 + \alpha_{13}^{d_2} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1), \\ Y(d_2, d_1, d_2) &= \alpha_{21}^{d_2} m_1 - \alpha_{22}^{d_1}(m_2 - 1) + \alpha_{23}^{d_2} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1), \\ Z(d_2, d_1, d_2) &= \alpha_{31}^{d_2} m_1 - \alpha_{32}^{d_1} m_2 + \alpha_{33}^{d_2}(m_3 - 1) + \epsilon_3(d_2) - \epsilon_3(d_1). \end{aligned}$$

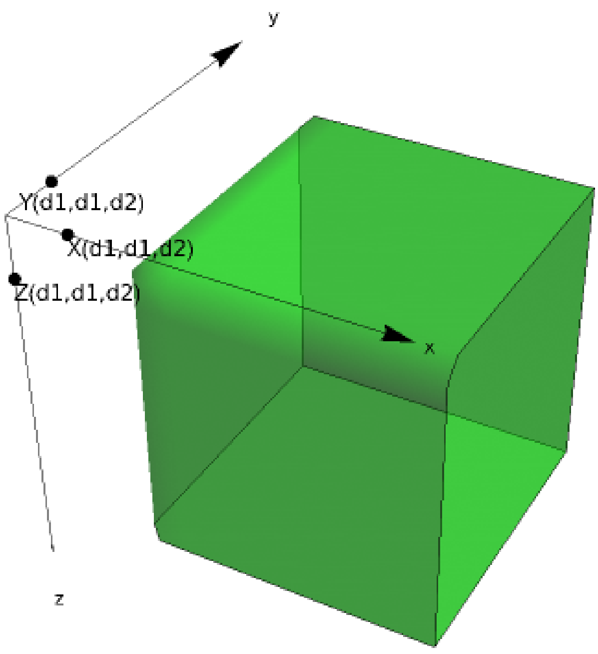


FIGURE 3.6: The united Nash equilibria region $\boldsymbol{NR}(d_1, d_1, d_2)$ in space.

Furthermore, the Nash region $\boldsymbol{NR}(d_2, d_1, d_2)$ is the sixth octant VI in Figure 3.1 and is given by

$$\begin{aligned}\boldsymbol{NR}(d_2, d_1, d_2) = \{(x, y, z) \in \mathbb{R}^3 \quad : \quad & x \leq X(d_2, d_1, d_2), \\ & y \geq Y(d_2, d_1, d_2), \\ & z \leq Z(d_2, d_1, d_2)\}.\end{aligned}$$

Hence, the united strategy (d_2, d_1, d_2) is a Nash equilibrium if and only if $(x, y, z) \in \boldsymbol{NR}(d_2, d_1, d_2)$ which is illustrated in Figure 3.7. The code used to generate this figure in Mathematica is shown in Appendix D.

The united strategy $(0, 0, 0)$ implies that all players of all types make decision d_2, so we may refer to this strategy by (d_2, d_2, d_2). Also, for simplicity to follow our model we denote to the corresponding horizontal threshold by $X(d_2, d_2, d_2)$, vertical threshold by $Y(d_2, d_2, d_2)$ and depth threshold by $Z(d_2, d_2, d_2)$. Using

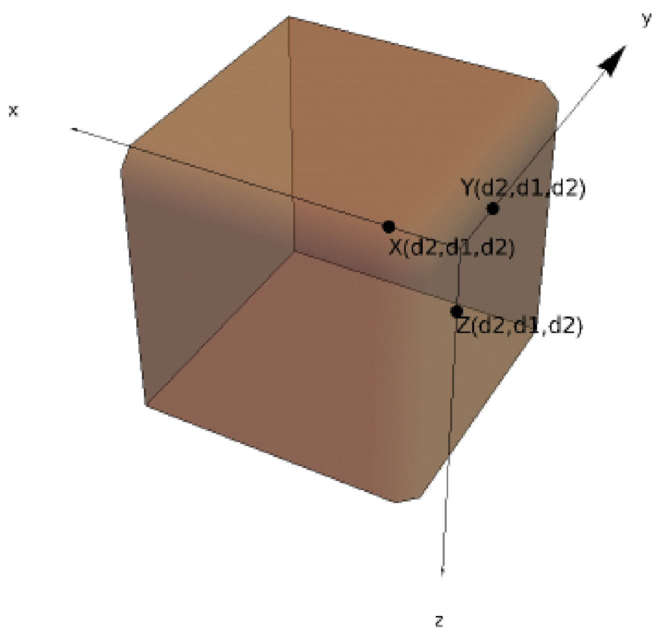


FIGURE 3.7: The united Nash equilibria region $\boldsymbol{NR}(d_2, d_1, d_2)$ in space.

Theorem 3.12, these thresholds can be writing as

$$
\begin{aligned}
X(d_2, d_2, d_2) &= \alpha_{11}^{d_2}(m_1 - 1) + \alpha_{12}^{d_2} m_2 + \alpha_{13}^{d_2} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1), \\
Y(d_2, d_2, d_2) &= \alpha_{21}^{d_2} m_1 + \alpha_{22}^{d_2}(m_2 - 1) + \alpha_{23}^{d_2} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1), \\
Z(d_2, d_2, d_2) &= \alpha_{31}^{d_2} m_1 + \alpha_{32}^{d_2} m_2 + \alpha_{33}^{d_2}(m_3 - 1) + \epsilon_3(d_2) - \epsilon_3(d_1).
\end{aligned}
$$

Furthermore, the Nash region $\boldsymbol{NR}(d_2, d_2, d_2)$ includes the seventh octant VII in Figure 3.1 and is given by

$$
\begin{aligned}
\boldsymbol{NR}(d_2, d_2, d_2) = \{(x, y, z) \in \mathbb{R}^3 \quad : \quad & x \leq X(d_2, d_2, d_2), \\
& y \leq Y(d_2, d_2, d_2), \\
& z \leq Z(d_2, d_2, d_2)\}.
\end{aligned}
$$

Hence, the united strategy (d_2, d_2, d_2) is a Nash equilibrium if and only if $(x, y, z) \in \boldsymbol{NR}(d_2, d_2, d_2)$ which is illustrated in Figure 3.8. The code used to generate this figure in Mathematica is shown in Appendix D.

The united strategy $(m_1, 0, 0)$ implies that all players of type t_1 make the decision d_1, but all players of types t_2 and t_3 makes the decision d_2, so we may refer to

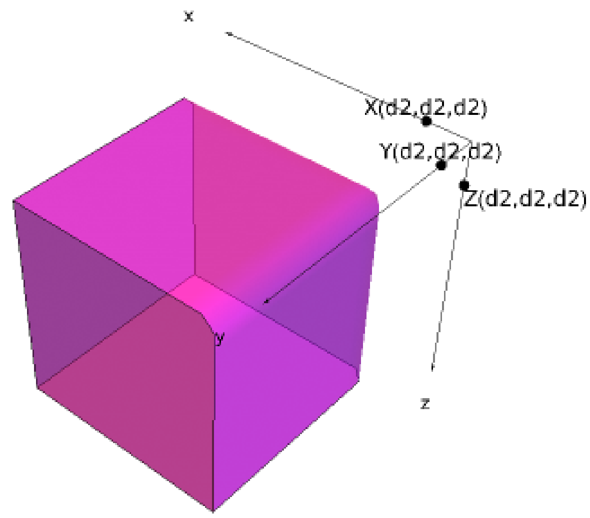


FIGURE 3.8: The united Nash equilibria region $\boldsymbol{NR}(d_2, d_2, d_2)$ in space.

this strategy by (d_1, d_2, d_2). Also, for simplicity to follow our model we denote to the correspondingly horizontal threshold by $X(d_1, d_2, d_2)$, vertical threshold by $Y(d_1, d_2, d_2)$ and depth threshold by $Z(d_1, d_2, d_2)$. Using Theorem 3.12, these thresholds can be writing as

$$\begin{aligned}
X(d_1, d_2, d_2) &= -\alpha_{11}^{d_1}(m_1 - 1) + \alpha_{12}^{d_2} m_2 + \alpha_{13}^{d_2} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1), \\
Y(d_1, d_2, d_2) &= -\alpha_{21}^{d_1} m_1 + \alpha_{22}^{d_2}(m_2 - 1) + \alpha_{23}^{d_2} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1), \\
Z(d_1, d_2, d_2) &= -\alpha_{31}^{d_1} m_1 + \alpha_{32}^{d_2} m_2 + \alpha_{33}^{d_2}(m_3 - 1) + \epsilon_3(d_2) - \epsilon_3(d_1).
\end{aligned}$$

Furthermore, the Nash region $\boldsymbol{NR}(d_1, d_2, d_2)$ includes the eighth octant $VIII$ in Figure 3.1 and is given by

$$\begin{aligned}
\boldsymbol{NR}(d_1, d_2, d_2) = \{(x, y, z) \in \mathbb{R}^3 \quad : \quad & x \geq X(d_1, d_2, d_2), \\
& y \leq Y(d_1, d_2, d_2), \\
& z \leq Z(d_1, d_2, d_2)\}.
\end{aligned}$$

Hence, the united strategy (d_1, d_2, d_2) is a Nash equilibrium if and only if $(x, y, z) \in \boldsymbol{NR}(d_1, d_2, d_2)$ which is illustrated in Figure 3.9. The code used to generate this

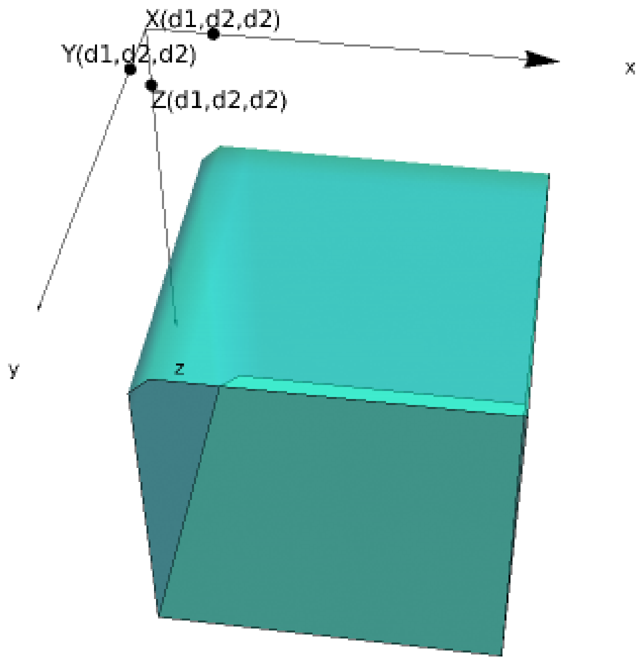


FIGURE 3.9: The united Nash equilibria region $\boldsymbol{NR}(d_1, d_2, d_2)$ in space.

figure in Mathematica is shown in Appendix D.

If we combined the previous eight united Nash regions, which means combining Figures (3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8 and 3.9) together, we will get a huge number of overlaping possibilities. We will show here some interesting possible of ordering the strategic thresholds.

The first figure we will show when the eight Nash regions are completely separated, i.e. there is no overlap between any two Nash regions. As we see in Figure 3.10 and Figure 3.11, there are eight Nash regions, each region clarify the Nash region for one possibility in a model of three types of players. That is, in any Nash region, we have only one united strategy that is Nash Equilibrium.

Note that we generate Figures 3.10 and 3.11 using Mathematica program which is shown in Appendix D, and we identify the axes: x, y and z. We collect the eight octants, where each octant represents Nash region for a certain Equilibrium as it was introduced in the beginnings of this section.

In these two figures, note that:

- Yellow color represent Nash region for (d_1, d_1, d_1) strategy;

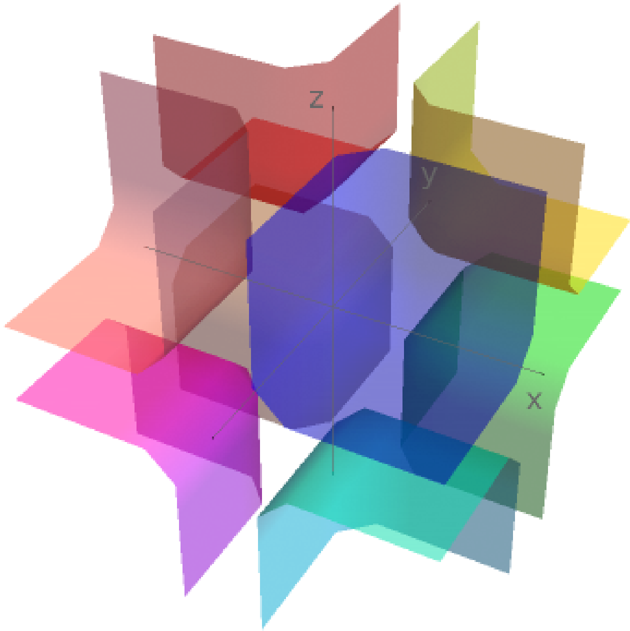


FIGURE 3.10: Degenerate united Nash equilibria regions when $A_{kk} < 0$ for all $k = 1, 2, 3$.

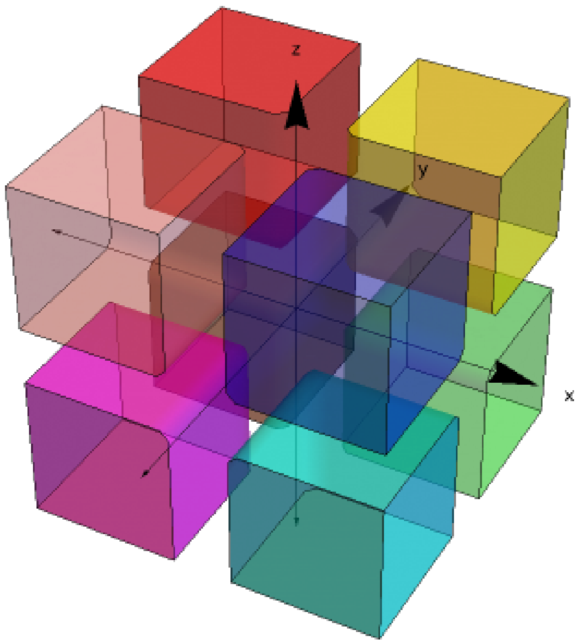
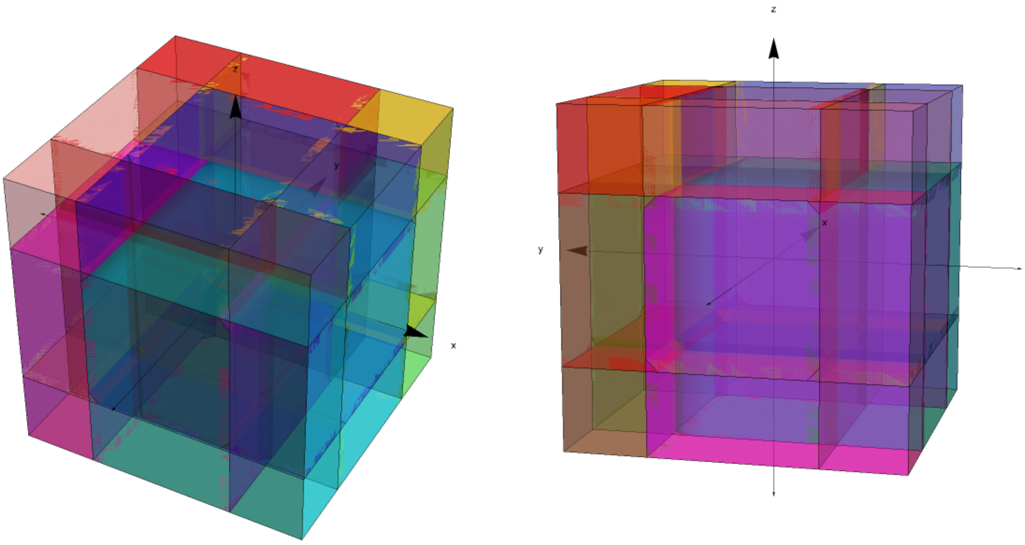


FIGURE 3.11: Non overlapping equilibria of Figure 3.10 when $A_{kk} < 0$ for all $k = 1, 2, 3$.



(a) Location A (b) Location B

FIGURE 3.12: Complex united Nash equilibria region when $A_{kk} > 0$ for all $k = 1, 2, 3$.

- Red color represent Nash region for (d_2, d_1, d_1) strategy;
- Orange color represent Nash region for (d_2, d_2, d_1) strategy;
- Blue color represent Nash region for (d_1, d_2, d_1) strategy;
- Green color represent Nash region for (d_1, d_1, d_2) strategy;
- Brown color represent Nash region for (d_2, d_1, d_2) strategy;
- Magenta color represent Nash region for (d_2, d_2, d_2) strategy;
- Cyan color represent Nash region for (d_1, d_2, d_2) strategy.

Now will show another interesting possibility of ordering the strategic thresholds, when all Nash regions for all united strategies are overlapping.

In Figure 3.12 we show two different location for such complexity of overlapping between the eight Nash united regions. New colors will represents different possibilities of overlapping. There are regions with two, three, four, five, six, seven, eight Nash equilibria which suppose to have different colors. The Mathematica code of Figure 3.12 is shown in Appendix D

Example 3.1. *We consider a simple overlapping of united Nash equilibria regions as follows:*

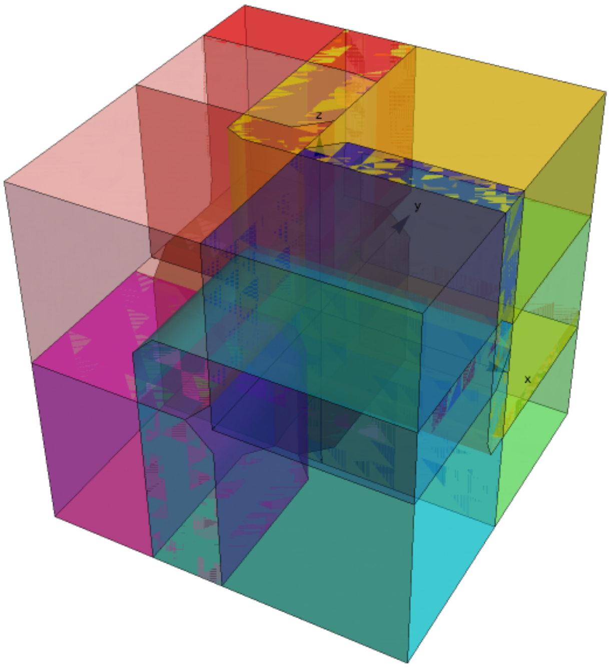


FIGURE 3.13: Simple overlapping of united Nash equilibria regions.

$$
\begin{aligned}
\boldsymbol{NR}(d_1,d_1,d_1) &= \{(x,y,z): x \geq -4, y \geq -2, z \geq -4\}, \\
\boldsymbol{NR}(d_2,d_1,d_1) &= \{(x,y,z): x \leq 0, y \geq 0, z \geq 0\}, \\
\boldsymbol{NR}(d_2,d_2,d_1) &= \{(x,y,z): x \leq 0, y \leq 6, z \geq 0\}, \\
\boldsymbol{NR}(d_1,d_2,d_1) &= \{(x,y,z): x \geq 0, y \leq 0, z \geq 0\}, \\
\boldsymbol{NR}(d_1,d_1,d_2) &= \{(x,y,z): x \geq -3, y \geq 0, z \leq 3\}, \\
\boldsymbol{NR}(d_2,d_1,d_2) &= \{(x,y,z): x \leq 0, y \geq 0, z \leq 0\}, \\
\boldsymbol{NR}(d_2,d_2,d_2) &= \{(x,y,z): x \leq 0, y \leq 0, z \leq 0\}, \\
\boldsymbol{NR}(d_1,d_2,d_2) &= \{(x,y,z): x \geq -4, y \leq 0, z \leq 4\}.
\end{aligned}
\tag{3.17}
$$

Figure 3.13 is an illustration of Example 3.1. Note that the value of x is represented on the horizontal x-axis, the value of y is represented on the vertical y-axis, and the value of z is represented on the depth z-axis.

The Mathematica code used to generate Figure 3.13 is shown in Appendix D. Note also that there are different overlaps between the united Nash equilibria represented in other colors that was used previously.

3.4 Separated Nash equilibria

The goal in this section is to determine and characterize all separated strategies that form Nash equilibria, by determining the necessary and sufficient conditions that guarantee the existence of separated Nash equilibria strategies.

Definition 3.15. *A separated strategy $S_D = (S_1, S_2, \cdots, S_n)$ is a pure strategy that is not united, where players of the same type tends to make different decisions.*

The separated strategy S_k is defined by

$$S_k : \boldsymbol{I}_k \longrightarrow \boldsymbol{D} \text{ for all } i_k \in \boldsymbol{I}_k.$$

such that $l_k \in \{1, 2, \cdots, m_{k-1}\}$ for at least $k \in \{1, \cdots, n\}$. Note that under the separated strategy $S_D \in \boldsymbol{S}$ we have $S_D(i_k) = d_1$ for some player i_k with type t_k or $S_D(i_k) = d_2$ for some player i_k with type t_k. Given $S_D \in \boldsymbol{S}$, assume the number of players who decide d_1 of type t_k is $l_k^{d_1}(S_D) = l_k$, and the number of players of type t_k who decide to make decision d_2 is $l_k^{d_2}(S_D) = m_k - l_k$.

Definition 3.16. *Let $A_{kj} = \alpha_{kj}^{d_1} + \alpha_{kj}^{d_2}$ for $k, j \in \{1, 2, \cdots, n\}$, be the coordinates of the influence crowding matrix. Hence, define $\boldsymbol{A}$ as*

$$\boldsymbol{A} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \in \mathbb{R}^{n \times n}.$$

Note that

- If $A_{kj} > 0$, then individuals with type t_j have a positive influence over the utility of the individuals with type t_k.
- If $A_{kj} = 0$, then individuals with type t_j are indifferent for the utility of the individuals with type t_k.

- If $A_{kj} < 0$, then individuals with type t_j have a negative influence over the utility of the individuals with type t_k.

Lemma 3.17. *Given strategy $S \in \boldsymbol{S}$. Let $(l_1, l_2, \cdots, l_n)$ be a Nash Equilibrium. If $A_{kk} > 0$, then $l_k \in \{0, m_k\}$, where $k \in \{1, \cdots, n\}$. Furthermore, if $A_{kk} > 0$ for all $k \in \{0, \cdots, n\}$, then $(l_1, l_2, \cdots, l_n)$ is united.*

Proof. (By contradiction). Assume $A_{kk} > 0$ and the strategy profile $(l_1, l_2, \cdots, l_n)$ is a Nash Equilibrium where $l_k \in \{1, 2, \cdots, m_k - 1\}$ for at least $k \in \{1, 2, \cdots, n\}$. Hence, the following inequalities hold

$$\Pi_k(d_1; l_1, l_2, \cdots, l_k, \cdots, l_n) \geq \Pi_k(d_2; l_1, \cdots, l_k - 1, \cdots, l_n) \text{ and} \quad (3.18)$$

$$\Pi_k(d_2; l_1, l_2, \cdots, l_k, \cdots, l_n) \geq \Pi_k(d_1; l_1, \cdots, l_k + 1, \cdots, l_n). \quad (3.19)$$

Substitute the utilities (3.5) and (3.6) in (3.18), we get

$$\omega_k^{d_1} + \alpha_{kk}^{d_1}(l_k - 1) + \sum_{j=1, j\neq k}^{n} \alpha_{kj}^{d_1} l_j + \epsilon_k(d_1) \geq \omega_k^{d_2} + \alpha_{kk}^{d_2}(m_k - l_k) + \sum_{j=1, j\neq k}^{n} \alpha_{kj}^{d_2}(m_j - l_j) + \epsilon_k(d_2). \quad (3.20)$$

And substitute the utilities (3.5) and (3.6) in (3.19), we get

$$\omega_k^{d_2} + \alpha_{kk}^{d_2}(m_k - l_k - 1) + \sum_{j=1, j\neq k}^{n} \alpha_{kj}^{d_2}(m_j - l_j) + \epsilon_k(d_2) \geq \omega_k^{d_1} + \alpha_{kk}^{d_1} l_k + \sum_{j=1, j\neq k}^{n} \alpha_{kj}^{d_1} l_j + \epsilon_k(d_1). \quad (3.21)$$

Combine two inequalities (3.20) and (3.21), we get

$$-\alpha_{kk}^{d_1} - \alpha_{kk}^{d_2} \geq 0,$$

which gives $A_{kk} \leq 0$. Hence, this contradicts the assumption of having A_{kk} is positive. Hence, Lemma (3.17) is hold. □

As we said before, the separated strategy is a strategy where the players split into two groups to make different decisions. So, the separated strategy requires that

$l_k \in \{1, 2, \cdots, m_k - 1\}$ for some $k \in \{1, 2, \cdots, n\}$.

Note that under the separated strategy S_D, we have $S_D(i_k) = d_1$ for some players of type t_k and $S_D(i_k) = d_2$ for some players of type t_k for at least one type t_k.

Let $\overline{\boldsymbol{S}_D}$ be the space of all separated strategies. Now, we will introduce a lemma for possible separated strategies, and show some examples on it.

Lemma 3.18. *Given n types of homogeneous players. The total number of separated strategies is*

$$\prod_{k=1}^{n}(m_k + 1) - 2^n,$$

where m_k is the number of players of type t_k.

Proof. By Induction.
Let $\overline{\boldsymbol{S}_D}$ be the space of all separated strategies. The space of all pure strategies equal $\overline{\boldsymbol{S}_D} \cup \overline{\boldsymbol{S}_U}$, where $\overline{\boldsymbol{S}_U}$ is the space of all united strategies.

$$\begin{aligned}
\text{If } n = 1 &\implies S_D \in \overline{\boldsymbol{S}_D} \\
\text{Under } S_D &\implies l_1 \in \{1, 2, \cdots, m_1 - 1\} \\
&\implies \text{So, there are } m_1 - 1 \text{ possibilities for the separated} \\
&\qquad \text{strategies which is equivalent to} \\
&\qquad \prod_{k=1}^{1}(m_k + 1) - 2^1.
\end{aligned}$$

Assume it is true for $n-1$ types of homogeneous players[3]. Then the total number of possible separated strategies will be

$$\prod_{k=1}^{n-1}(m_k+1)-2^{n-1}.$$

Now, for n types of homogeneous players, it follows that by Lemma 3.3 the total number of separated strategies is

$$\prod_{k=1}^{n-1+1}(m_k+1)-(2^{n-1}+2^{n-1}).$$

This is because adding one more type includes 2^{n-1} cohesive strategies by Lemma 3.9. The last identity is equivalent to

$$\prod_{k=1}^{n}(m_k+1)-2^{n},$$

which ends the proof. □

Example 3.2. *[15] Let $n=1$, and we have m players. Then there are $m-1$ possible separated strategies, where $S_D \in \{1,2,\cdots,m-1\}$. Otherwise, the strategy will be united strategy as we introduced in Section 3.2, where $l_1 \in \{0,m\}$. In Table 3.1 we show that we have $m+1$ pure strategies, strategies in gray cells are united and the rest are separated.*

TABLE 3.1: Pure strategies for 1 type of players.

Type 1	0	1	2	...	m

[3]If $n=2 \Longrightarrow S_D \in \overline{\boldsymbol{S}_D}$
Under $S_D \Longrightarrow \{l_1 \in \{1,2,\cdots,m_1-1\}$ and $l_2 \in \{0,1,2,\cdots,m_2\}\}$
Or $\{l_1 \in \{0,1,2,\cdots,m_1\}$ and $l_2 \in \{1,2,\cdots,m_2-1\}\}$
$\Longrightarrow$ So, there are $m_1m_2+m_1+m_2-3$ possibilities for the separated strategies which is equivalent to

$$\prod_{k=1}^{2}(m_k+1)-2^{2}.$$

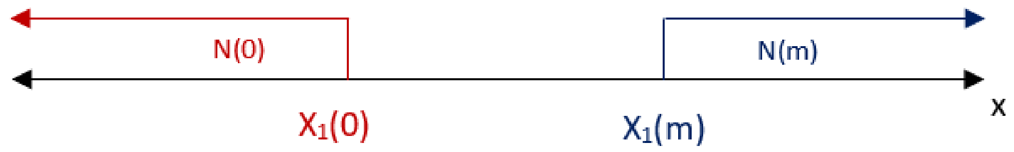


FIGURE 3.14: Pure Nash equilibria interval when $A < 0$.

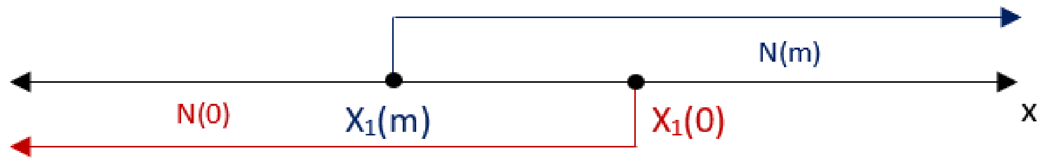


FIGURE 3.15: United Nash equilibria interval when $A > 0$.

We can see in Figure 3.14 the Nash intervals for pure strategies, where the right interval is for united strategies given by

$$x > X_1(m) = -\alpha_{11}^{d_1}(m-1).$$

This threshold represents the case where all players make decision d_1. The left interval is for united strategies given by

$$x < X_1(0) = \alpha_{11}^{d_2}(m-1).$$

This threshold represents the case where all players make decision d_2. While the middle interval is for separated strategies.

Example 3.3. *Let $n = 2$, which means we have two types t_1 and t_2 with m_1 and m_2 players, respectively. We present the pure strategies using the Table 3.2, which refers to all possible separated and united strategies.*
In Table 3.2 we show all possible pure strategies for two types of players which equal $(m_1+1)\cdot(m_2+1) = m_1m_2+m_1+m_2+1$, and we can see that the corners represent the united strategies which equal (as we saw before by Lemma 3.9) $2^2 = 4$. So, we can see that the number of possible separated strategies for two types of players equal

$$m_1m_2 + m_1 + m_2 + 1 - 4 = m_1m_2 + m_1 + m_2 - 3,$$

where m_1 represent the number of players of type t_1 and m_2 represents the number of players of type t_2.

Table 3.2: Pure strategies for 2 types of players

		Type 2				
Type 1	$l_1 \backslash l_2$	0	1	...	$m_2 - 1$	m_2
	0	(0,0)	(0,1)	...	$(0, m_2 - 1)$	$(0,m_2)$
	1	(1,0)	(1,1)	...	$(1, m_2 - 1)$	$(1,m_2)$
	2	(2,0)	(2,1)	...	$(2, m_2 - 1)$	$(2,m_2)$
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
	$m_1 - 1$	$(m_1 - 1,0)$	$(m_1 - 1,1)$	...	$(m_1 - 1, m_2 - 1)$	$(m_1 - 1,m_2)$
	m_1	$(m_1,0)$	$(m_1,1)$	...	$(m_1, m_2 - 1)$	(m_1,m_2)

Example 3.4. *Let $n = 2$, where there are two type of players. Let $m_1 = m_2 = 4$, i.e. the number of players for each type are four players. Let $A_{11} < 0$ and $A_{22} > 0$, since A_{11} not greater than zero, then $l_1 \in \{0, 1, 2, 3, 4\}$. But, $l_2 \in \{0, 4\}$ since $A_{22} > 0$. See Figure 3.16 which clarify the Nash domains for this game.*
As we can see in the figure, the Nash domain doesn't include strategies where $l_2 \in \{1, 2, 3\}$, since as we said before $A_{22} > 0$.
The blue regions represents two united strategies that are Nash Equilibrium. The graph also represent the thresholds for united strategies. The areas in light violet colour, represent the Nash domain for separated strategies, where $l_1 \in \{1, 2, 3\}$ and $l_2 \in \{0, 4\}$.

Example 3.5. *Let $n = 3$, and we have m_1, m_2, m_3 players of type t_1, t_2, t_3 respectively. We will represent the pure strategies using the following Table 3.3, which refers to all possible pure (united and separated) strategies, which equal*

$$(m_1+1)\cdot(m_2+1)\cdot(m_3+1) = m_1m_2m_3+m_1m_2+m_1m_3+m_2m_3+m_3+m_2+m_1+1.$$

The gray cells reflex the united strategies which equal to $2^3 = 8$ according to Lemma 3.9. So, the number of possibilities of separated strategies for a game contains three types of people equal

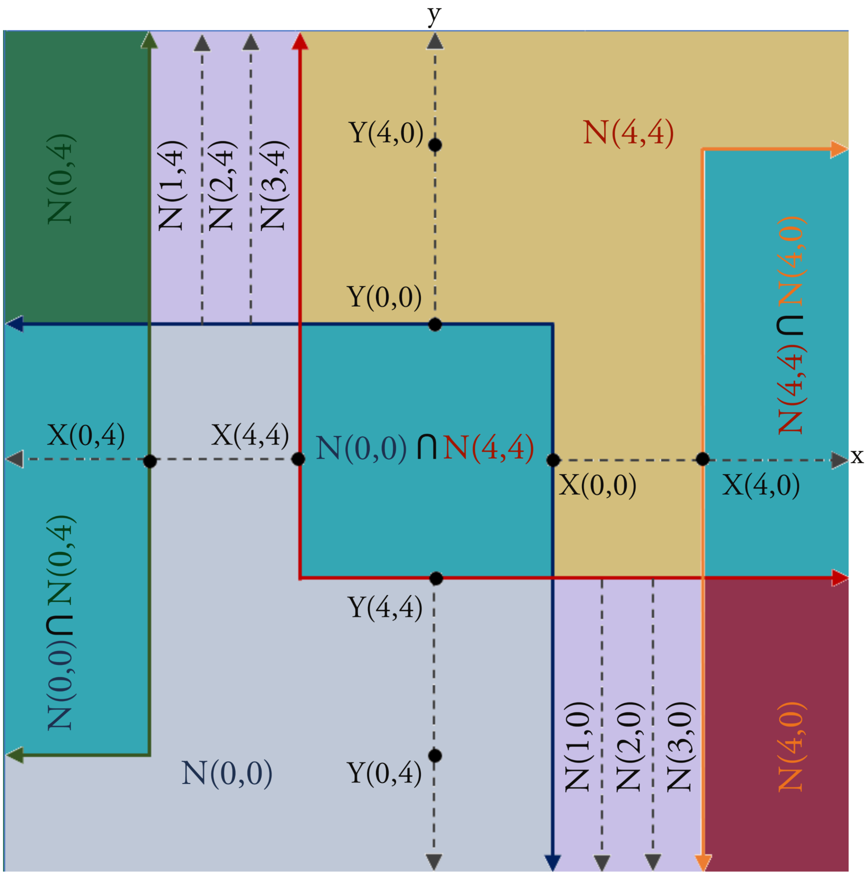


FIGURE 3.16: Separated Nash domain $N(l_1, l_2)$ when $A_{11} < 0, A_{12} < 0, A_{21} > 0, A_{22} > 0$ and $m_1 = m_2 = 4$.

$$m_1m_2m_3 + m_1m_2 + m_1m_3 + m_2m_3 + m_3 + m_2 + m_1 + 1 - 8 =$$
$$m_1m_2m_3 + m_1m_2 + m_1m_3 + m_2m_3 + m_3 + m_2 + m_1 - 7,$$

where m_1, m_2, m_3 reflects numbers of players of type t_1, t_2, t_3 respectively.

Let C be a vector in $\mathbb{R}^n$ defined by

$$C = \Big(X_1(0, 0, \cdots, 0), X_2(0, 0, \cdots, 0), \cdots, X_n(0, 0, \cdots, 0)\Big).$$

Definition 3.19. *Define the united k^{th} vector $\vec{X_k}$ by*

$$\begin{aligned} \vec{X_k} \quad &= \quad \big(X_1(0, \cdots, 0, m_k, 0, \cdots, 0), X_2(0, \cdots, 0, m_k, 0, \cdots, 0), \cdots, \\ &\qquad X_n(0, \cdots, 0, m_k, 0, \cdots, 0)\big) - C \end{aligned} \tag{3.22}$$

Table 3.3: Pure strategies for 3 types of players

Type 1	Type 2	Type 3				
		0	1	...	m_3-1	m_3
0	0	(0,0,0)	(0,0,1)	...	(0,0,m_3-1)	(0,0,m_3)
	1	(0,1,0)	(0,1,1)	...	(0,1,m_3-1)	(0,1,m_3)
	2	(0,2,0)	(0,2,1)	...	(0,2,m_3-1)	(0,2,m_3)
	⋮	⋮	⋮		⋮	⋮
	m_2	(0,m_2,0)	(0,m_2,1)	...	(0,m_2,m_3-1)	(0,m_2,m_3)
1	0	(1,0,0)	(1,0,1)	...	(1,0,m_3-1)	(1,0,m_3)
	1	(1,1,0)	(1,1,1)	...	(1,1,m_3-1)	(1,1,m_3)
	2	(1,2,0)	(1,2,1)	...	(1,2,m_3-1)	(1,2,m_3)
	⋮	⋮	⋮		⋮	⋮
	m_2	(1,m_2,0)	(1,m_2,1)	...	(1,m_2,m_3-1)	(1,m_2,m_3)
⋮	⋮	⋮	⋮		⋮	⋮
⋮	⋮	⋮	⋮		⋮	⋮
m_1	0	(m_1,0,0)	(m_1,0,1)	...	(m_1,0,m_3-1)	(m_1,0,m_3)
	1	(m_1,1,0)	(m_1,1,1)	...	(m_1,1,m_3-1)	(m_1,1,m_3)
	2	(m_1,2,0)	(m_1,2,1)	...	(m_1,2,m_3-1)	(m_1,2,m_3)
	⋮	⋮	⋮		⋮	⋮
	m_2	(m_1,m_2,0)	(m_1,m_2,1)	...	(m_1,m_2,m_3-1)	(m_1,m_2,m_3)

Lemma 3.20. *One can show that $\vec{X_k}$ in 3.19 can be written as*

$$\vec{X_k} = -(A_{1k}m_k, A_{2k}m_k, \cdots, A_{k-1k}m_k, A_{kk}(m_k-1), A_{k+1k}m_k, \cdots, A_{nk}m_k).$$

Proof. We know from Definition 3.19

$$\begin{aligned} \vec{X_k} &= \big(X_1(0,\cdots,0,m_k,0,\cdots,0), X_2(0,\cdots,0,m_k,0,\cdots,0),\cdots, \\ &\quad X_n(0,\cdots,0,m_k,0,\cdots,0)\big) - C, \end{aligned} \tag{3.23}$$

by substituting in Theorem 3.12 we get

$$\vec{X}_k = \Bigg(\Big(-\alpha_{1k}^{d_1} m_k + \sum_{\substack{j\in N_2\\ j\neq 1\\ j\neq k}} \alpha_{1j}^{d_2} m_j + \alpha_{11}^{d_2}(m_1-1) - \big(\sum_{\substack{j\in N_2\\ j\neq 1}} \alpha_{1j}^{d_2} m_j + \alpha_{11}^{d_2}(m_1-1)\big)\Big),$$

$$\Big(-\alpha_{2k}^{d_1} m_k + \sum_{\substack{j\in N_2\\ j\neq 2\\ j\neq k}} \alpha_{2j}^{d_2} m_j + \alpha_{22}^{d_2}(m_2-1) - \big(\sum_{\substack{j\in N_2\\ j\neq 2}} \alpha_{2j}^{d_2} m_j + \alpha_{22}^{d_2}(m_2-1)\big)\Big), \cdots,$$

$$\Big(-\alpha_{kk}^{d_1} (m_k-1) + \sum_{\substack{j\in N_2\\ j\neq k}} \alpha_{kj}^{d_2} m_j - \big(\sum_{\substack{j\in N_2\\ j\neq k}} \alpha_{kj}^{d_2} m_j + \alpha_{kk}^{d_2}(m_k-1)\big)\Big),$$

$$\Big(-\alpha_{k+1k}^{d_1} m_k + \sum_{\substack{j\in N_2\\ j\neq k+1\\ j\neq k}} \alpha_{k+1j}^{d_2} m_j + \alpha_{k+1k+1}^{d_2}(m_{k+1}-1) - \big(\sum_{\substack{j\in N_2\\ j\neq k+1}} \alpha_{k+1j}^{d_2} m_j$$

$$+ \alpha_{k+1k+1}^{d_2}(m_{k+1}-1)\big)\Big), \cdots,$$

$$\Big(-\alpha_{nk}^{d_1} m_k + \sum_{\substack{j\in N_2\\ j\neq n\\ j\neq k}} \alpha_{nj}^{d_2} m_j + \alpha_{nn}^{d_2}(m_n-1) - \big(\sum_{\substack{j\in N_2\\ j\neq n}} \alpha_{nj}^{d_2} m_j + \alpha_{nn}^{d_2}(m_n-1)\big)\Big)\Bigg).$$

By rearranging the terms we get

$$\begin{aligned} \vec{X}_k &= \Bigg(\Big(-\alpha_{1k}^{d_1} m_k - \alpha_{1k}^{d_2} m_k\Big), \Big(-\alpha_{2k}^{d_1} m_k - \alpha_{2k}^{d_2} m_k\Big), \cdots, \\ &\quad \Big(-\alpha_{kk}^{d_1} (m_k-1) - \alpha_{kk}^{d_2} (m_k-1)\Big), \\ &\quad \Big(-\alpha_{k+1k}^{d_1} m_k - \alpha_{k+1k}^{d_2} m_k\Big), \cdots, \Big(-\alpha_{nk}^{d_1} m_k - \alpha_{nk}^{d_2} m_k\Big)\Bigg). \end{aligned}$$

Now, by taking the common factor we get

$$\begin{aligned} \vec{X}_k &= \Bigg(\Big((-\alpha_{1k}^{d_1} - \alpha_{1k}^{d_2}) m_k\Big), \Big((-\alpha_{2k}^{d_1} - \alpha_{2k}^{d_2}) m_k\Big), \cdots, \Big((-\alpha_{kk}^{d_1} - \alpha_{kk}^{d_2})(m_k-1)\Big), \\ &\quad \Big((-\alpha_{k+1k}^{d_1} - \alpha_{k+1k}^{d_2}) m_k\Big), \cdots, \Big((-\alpha_{nk}^{d_1} - \alpha_{nk}^{d_2}) m_k\Big)\Bigg). \end{aligned}$$

By substituting in 3.16 we get

$$\vec{X}_k = -(A_{1k} m_k, A_{2k} m_k, \cdots, A_{k-1k} m_k, A_{kk}(m_k-1), A_{k+1k} m_k, \cdots, A_{nk} m_k),$$

which complete the proof. □

By Lemma 3.20, we note that

$$\vec{X}_1 = -(A_{11}(m_1 - 1), A_{21}m_1, \cdots, A_{n1}m_1),$$

$$\vec{X}_2 = -(A_{12}m_2, A_{22}(m_2 - 1), A_{32}m_2, \cdots, A_{n2}m_2),$$

and so on.

Definition 3.21. *The separated vector $\vec{Z}(l_1, l_2, \cdots, l_n)$ is defined by*

$$\begin{aligned}\vec{Z}(l_1, l_2, \cdots, l_n) = &- l_1(A_{11}, A_{21}, \cdots, A_{n1}) - l_2(A_{12}, A_{22}, \cdots, A_{n2}) - \cdots \\ &- l_n(A_{1n}, A_{2n}, \cdots, A_{nn}).\end{aligned}$$

Lemma 3.22. *One can show $\vec{Z}(l_1, l_2, \cdots, l_n)$ in 3.21 can be written as*

$$\begin{aligned}\vec{Z}(l_1, l_2, \cdots, l_n) \quad = \quad & \frac{l_1}{m_1 - 1}\big(\vec{X}_1 + (0, A_{21}, \cdots, A_{n1})\big) \\ & + \frac{l_2}{m_2 - 1}\big(\vec{X}_2 + (A_{12}, 0, A_{32}, \cdots, A_{n2})\big) + \cdots \\ & + \frac{l_n}{m_n - 1}\big(\vec{X}_n + (A_{1n}, A_{2n}, \cdots, A_{(n-1)n}, 0)\big).\end{aligned}$$

Proof. We know from Definition 3.21

$$\begin{aligned}\vec{Z}(l_1, l_2, \cdots, l_n) \quad = \quad & -l_1(A_{11}, A_{21}, \cdots, A_{n1}) - l_2(A_{12}, A_{22}, \cdots, A_{n2}) - \cdots \\ & -l_n(A_{1n}, A_{2n}, \cdots, A_{nn}), \\ = \quad & l_1(-A_{11}, -A_{21}, \cdots, -A_{n1}) + l_2(-A_{12}, -A_{22}, \cdots, -A_{n2}) \\ & + \cdots + l_n(-A_{1n}, -A_{2n}, \cdots, -A_{nn}),\end{aligned}$$

multiply each l_k by m_k and divide by m_k for all $k \in \{1, 2, \cdots, n\}$, we get

$$\begin{aligned}\vec{Z}(l_1, l_2, \cdots, l_n) &= \frac{l_1}{m_1}(-A_{11}m_1, -A_{21}m_1, \cdots, -A_{n1}m_1) \\ &+ \frac{l_2}{m_2}(-A_{12}m_2, -A_{22}m_2, \cdots, -A_{n2}m_2) + \cdots \\ &+ \frac{l_n}{m_n}(-A_{1n}m_n, -A_{2n}m_n, \cdots, -A_{nn}m_n),\end{aligned}$$

add m_k and substract m_k from $-A_{kk}m_k$, we get

$$\begin{aligned}\vec{Z}(l_1, l_2, \cdots, l_n) &= \frac{l_1}{m_1}(-A_{11}m_1 + A_{11} - A_{11}, -A_{21}m_1, \cdots, -A_{n1}m_1) \\ &+ \frac{l_2}{m_2}(-A_{12}m_2, -A_{22}m_2 + A_{22} - A_{22}, \cdots, -A_{n2}m_2) + \cdots \\ &+ \frac{l_n}{m_n}(-A_{1n}m_n, -A_{2n}m_n, \cdots, -A_{nn}m_n + A_{nn} - A_{nn}),\end{aligned}$$

take the common factor A_{kk} for all $k \in \{1, 2, \cdots, n\}$

$$\begin{aligned}\vec{Z}(l_1, l_2, \cdots, l_n) &= \frac{l_1}{m_1}(-A_{11}(m_1 - 1) - A_{11}, -A_{21}m_1, \cdots, -A_{n1}m_1) \\ &+ \frac{l_2}{m_2}(-A_{12}m_2, -A_{22}(m_2 - 1) - A_{22}, \cdots, -A_{n2}m_2) + \cdots \\ &+ \frac{l_n}{m_n}(-A_{1n}m_n, -A_{2n}m_n, \cdots, -A_{nn}(m_n - 1) - A_{nn}),\end{aligned}$$

rearrange using Lemma 3.20 we get

$$\begin{aligned}\vec{Z}(l_1, l_2, \cdots, l_n) &= \frac{l_1}{m_1}(\vec{X}_1 - (A_{11}, 0, 0, \cdots, 0)) + \frac{l_2}{m_2}(\vec{X}_2 - (0, A_{22}, 0, 0, \cdots, 0)) \\ &+ \cdots + \frac{l_n}{m_n}(\vec{X}_n - (0, 0, \cdots, 0, A_{nn})),\end{aligned} \tag{3.24}$$

add and subtract $(A_{1k}, A_{2k}, \cdots, A_{k-1k}, 0, A_{k+1k}, \cdots, A_{nk})$ to each element, then we have

$$\begin{aligned}
\vec{Z}(l_1, l_2, \cdots, l_n) &= \frac{l_1}{m_1}(\vec{X_1} + (0, A_{21}, \cdots, A_{n1}) - (A_{11}, A_{21}, \cdots, A_{n1})) \\
&+ \frac{l_2}{m_2}(\vec{X_2} + (A_{12}, 0, A_{32}, \cdots, A_{n2}) - (A_{12}, A_{22}, A_{32}, \cdots, A_{n2})) \\
&+ \cdots + \frac{l_n}{m_n}(\vec{X_n} + (A_{1n}, A_{2n}, \cdots, A_{(n-1)n}, 0) \\
&-(A_{1n}, A_{2n}, \cdots, A_{(n-1)n}, A_{nn})),
\end{aligned}$$

multiply by $m_k - 1$ and divide by $m_k - 1$ for all $k \in \{1, \cdots, n\}$ and rearrange the terms we get

$$\begin{aligned}
\vec{Z}(l_1, l_2, \cdots, l_n) &= \frac{l_1}{m_1 - 1}(\vec{X_1} + (0, A_{21}, \cdots, A_{n1})) \\
&+ \frac{l_2}{m_2 - 1}(\vec{X_2} + (A_{12}, 0, A_{32}, \cdots, A_{n2})) + \cdots \\
&+ \frac{l_n}{m_n - 1}(\vec{X_n} + (A_{1n}, A_{2n}, \cdots, A_{(n-1)n}, 0)),
\end{aligned}$$

which completes the proof. □

Theorem 3.23. *Let $l_k \in \{1, 2, \cdots, m_k - 1\}$ for all $k = 1, 2, \cdots, n$.*

- *If $A_{kk} \leq 0$, then the separated Nash region $\boldsymbol{NR}(0, 0, \cdots, 0, l_k, 0 \cdots, 0)$ is given by*

$$\begin{aligned}
&\boldsymbol{NR}(0, 0, \cdots, 0, l_k, 0 \cdots, 0) = \big\{C + \vec{Z}(0, 0, \cdots, 0, l_k, 0 \cdots, 0) \\
&+ (p_1, p_2, \cdots, p_{k-1}, p_k A_{kk}, p_{k+1}, \cdots, p_n) : \\
&p_k \in [0, 1],\ p_j \in (-\infty, 0]\ \forall j \in \{1, 2, \cdots, k-1, k+1, \cdots, n\}\big\}.
\end{aligned}$$

and the separated Nash region $\boldsymbol{NR}(m_1, m_2, \cdots, m_{k-1}, l_k, m_{k+1}, \cdots, m_n)$ *is given by*

$$\boldsymbol{NR}(m_1, m_2, \cdots, m_{k-1}, l_k, m_{k+1}, \cdots, m_n) = \big\{C$$
$$+ \vec{Z}(m_1 - 1, m_2 - 1, \cdots, m_{k-1} - 1, l_k, m_{k+1} - 1, \cdots, m_n - 1)$$
$$+ (p_1, p_2, \cdots, p_{k-1}, p_k A_{kk}, p_{k+1}, \cdots, p_n) :$$
$$p_k \in [0, 1],\ p_j \in [0, \infty)\ \forall j \in \{1, 2, \cdots, k-1, k+1, \cdots, n\}\big\}.$$

- *If* $A_{kk} \leq 0\ \ \forall k \in \{1, 2, \cdots, n\}$*, then the separated Nash region* $\boldsymbol{NR}(l_1, l_2, \cdots, l_n)$ *is given by*

$$\boldsymbol{NR}(l_1, l_2, \cdots, l_n) = \big\{C + \vec{Z}(l_1, l_2, \cdots, l_n) +$$
$$(p_1 A_{11}, p_2 A_{22}, \cdots, p_n A_{nn}) : p_k \in [0, 1]\ \forall k \in \{1, 2, \cdots, n\}\big\}.$$

Hence, if the individuals with a given type have a non-positive influence over the utility of the individuals with the same type, i.e. $A_{kk} \leq 0$ *for all* $k \in \{1, 2, \cdots, n\}$*, then for every* $(l_1, l_2, \cdots, l_n)$ *separated strategic set there are relative preferences for which* $(l_1, l_2, \cdots, l_n)$ *is a Nash equilibrium set.*

Proof. The $(0, \cdots, 0, l_k, 0, \cdots, 0)$ strategy is a Nash equilibrium if and only if the following inequalities hold

$$\Pi_k(d_1; 0, \cdots, 0, l_k, 0, \cdots, 0) \geq \Pi_k(d_2; 0, \cdots, 0, l_k - 1, 0, \cdots, 0), \tag{3.25}$$
$$\Pi_k(d_2; 0, \cdots, 0, l_k, 0, \cdots, 0) \geq \Pi_k(d_1; 0, \cdots, 0, l_k + 1, 0, \cdots, 0), \tag{3.26}$$
$$\Pi_j(d_2; 0, \cdots, 0, l_k, 0, \cdots, 0) \geq \Pi_j(d_1; 0, \cdots, 0, 1_j, 0, \cdots, 0, l_k, 0, \cdots, 0) \tag{3.27}$$

$\forall j \in \{1, 2, \cdots, k-1, k+1, \cdots, n\}$, where 1_j refer to value equal 1 in position j.

Now substitute the utility functions from (3.5) and (3.6) in inequality (3.25) we obtain

$$\omega_k^{d_1} + \alpha_{k1}^{d_1}0 + \alpha_{k2}^{d_1}0 + \cdots + \alpha_{kk-1}^{d_1}0 + \alpha_{kk}^{d_1}(l_k - 1) + \alpha_{kk+1}^{d_1}0 + \cdots + \alpha_{kn}^{d_1}0 + \epsilon_k(d_1) \geq$$
$$\omega_k^{d_2} + \alpha_{k1}^{d_2}(m_1 - 0) + \alpha_{k2}^{d_2}(m_2 - 0) + \cdots + \alpha_{kk-1}^{d_2}(m_{k-1} - 0)$$
$$+ \alpha_{kk}^{d_2}(m_k - (l_k - 1) - 1) + \alpha_{kk+1}^{d_2}(m_{k+1} - 0) + \cdots + \alpha_{kn}^{d_2}(m_n - 0) + \epsilon_k(d_2),$$

Rearranging the terms in the above inequality we get

$$\omega_k^{d_1} - \omega_k^{d_2} \geq -\alpha_{kk}^{d_1}(l_k - 1) + \alpha_{k1}^{d_2}m_1 + \alpha_{k2}^{d_2}m_2 + \cdots + \alpha_{kk-1}^{d_2}m_{k-1} + \alpha_{kk}^{d_2}(m_k - l_k)$$
$$+ \alpha_{kk+1}^{d_2}m_{k+1} + \cdots + \alpha_{kn}^{d_2}m_n + \epsilon_k(d_2) - \epsilon_k(d_1),$$

which can simplify to

$$x_k \geq -\alpha_{kk}^{d_1}(l_k - 1) - \alpha_{kk}^{d_2}(l_k - 1) + \alpha_{k1}^{d_2}m_1 + \alpha_{k2}^{d_2}m_2 + \cdots + \alpha_{kk-1}^{d_2}m_{k-1}$$
$$+ \alpha_{kk}^{d_2}(m_k - 1) + \alpha_{kk+1}^{d_2}m_{k+1} + \cdots + \alpha_{kn}^{d_2}m_n + \epsilon_k(d_2) - \epsilon_k(d_1).$$

Using the X_k threshold $X_k(0, 0, \cdots, 0)$ strategy profile 3.14 we get

$$x_k \geq X_k(0, 0, \cdots, 0) - A_{kk}(l_k - 1) + \epsilon_k(d_2) - \epsilon_k(d_1),$$

Now substitute the utility functions from (3.5) and (3.6) in inequality (3.26) we obtain

$$\omega_k^{d_2} + \alpha_{k1}^{d_2}(m_1 - 0) + \alpha_{k2}^{d_2}(m_2 - 0) + \cdots + \alpha_{kk-1}^{d_2}(m_{k-1} - 0) + \alpha_{kk}^{d_2}(m_k - l_k - 1)$$
$$+ \alpha_{kk+1}^{d_2}(m_{k+1} - 0) + \cdots + \alpha_{kn}^{d_2}(m_n - 0) + \epsilon_k(d_2) \geq \omega_k^{d_1} + \alpha_{k1}^{d_1}0 + \alpha_{k2}^{d_1}0 + \cdots$$
$$+ \alpha_{kk-1}^{d_1}0 + \alpha_{kk}^{d_1}(l_k + 1 - 1) + \alpha_{kk+1}^{d_1}0 + \cdots + \alpha_{kn}^{d_1}0 + \epsilon_k(d_1)$$

Rearranging the terms in the above inequality we get

$$\omega_k^{d_1} - \omega_k^{d_2} \leq -\alpha_{kk}^{d_1}l_k + \alpha_{k1}^{d_2}m_1 + \alpha_{k2}^{d_2}m_2 + \cdots + \alpha_{kk-1}^{d_2}m_{k-1} + \alpha_{kk}^{d_2}(m_k - l_k - 1)$$
$$+ \alpha_{kk+1}^{d_2}m_{k+1} + \cdots + \alpha_{kn}^{d_2}m_n + \epsilon_k(d_2) - \epsilon_k(d_1).$$

Using the X_k threshold $X_k(0,0,\cdots,0)$ strategy profile 3.14 we get

$$x_k \leq -\alpha_{kk}^{d_1} l_k - \alpha_{kk}^{d_2} l_k + X_k(0,0,\cdots,0) + \epsilon_k(d_2) - \epsilon_k(d_1),$$

which can simplify to

$$x_k \leq X_k(0,0,\cdots,0) - A_{kk} l_k + \epsilon_k(d_2) - \epsilon_k(d_1).$$

Now substitute the utility functions from (3.5) and (3.6) in inequality (3.27) we obtain

$$\begin{aligned}
&\omega_j^{d_2} + \alpha_{j1}^{d_2}(m_1 - 0) + \alpha_{j2}^{d_2}(m_2 - 0) + \cdots + \alpha_{jj-1}^{d_2}(m_{j-1} - 0) + \alpha_{jj}^{d_2}(m_j - 0 - 1)\\
&+ \alpha_{jj+1}^{d_2}(m_{j+1} - 0) + \cdots + \alpha_{jk-1}^{d_2}(m_{k-1} - 0) + \alpha_{jk}^{d_2}(m_k - l_k) + \alpha_{jk+1}^{d_2}(m_{k+1} - 0)\\
&+ \cdots + \alpha_{jn}^{d_2}(m_n - 0) + \epsilon_j(d_2) \geq \omega_j^{d_1} + \alpha_{j1}^{d_1} 0 + \alpha_{j2}^{d_1} 0 + \cdots + \alpha_{jj-1}^{d_1} 0 + \alpha_{jj}^{d_1}(1 - 1)\\
&+ \alpha_{jj+1}^{d_1} 0 + \cdots + \alpha_{jk-1}^{d_1} 0 + \alpha_{jk}^{d_1} l_k + \alpha_{jk+1}^{d_1} 0 + \cdots + \alpha_{jn}^{d_1} 0 + \epsilon_j(d_1),
\end{aligned}$$

Rearranging the terms in the above inequality we get

$$\begin{aligned}
&\omega_j^{d_1} - \omega_j^{d_2} \leq -\alpha_{jk}^{d_1} l_k + \alpha_{j1}^{d_2} m_1 + \alpha_{j2}^{d_2} m_2 + \cdots + \alpha_{jj-1}^{d_2} m_{j-1} + \alpha_{jj}^{d_2}(m_j - 1)\\
&+ \alpha_{jj+1}^{d_2} m_{j+1} + \cdots + \alpha_{jk-1}^{d_2} m_{k-1} + \alpha_{jk}^{d_2}(m_k - l_k) + \alpha_{jk+1}^{d_2} m_{k+1} + \cdots\\
&+ \alpha_{jn}^{d_2} m_n + \epsilon_j(d_2) - \epsilon_j(d_1).
\end{aligned}$$

Using the X_j threshold $X_j(0,0,\cdots,0)$ strategy profile 3.14 we get

$$x_j \leq -\alpha_{jk}^{d_1} l_k - \alpha_{jk}^{d_2} l_k + X_j(0,0,\cdots,0) + \epsilon_j(d_2) - \epsilon_j(d_1),$$

which can simplify to

$$x_j \leq X_j(0,0,\cdots,0) - A_{jk} l_k + \epsilon_j(d_2) - \epsilon_j(d_1).$$

We note that

$$
\begin{aligned}
& C+\vec{Z}(0,\cdots,0,l_k,0,\cdots,0)+(p_1,\cdots,p_{k-1},p_kA_{kk},p_{k+1},\cdots,p_n) \\
= & \big((x_1(0,\cdots,0),\cdots,(x_n(0,\cdots,0)\big)-l_k(A_{1k},\cdots,A_{nk}) \\
& +(p_1,\cdots,p_{k-1},p_kA_{kk},p_{k+1},\cdots,p_n), \\
= & \big(X_1(0,\cdots,0)-A_{1k}l_k,\cdots,X_n(0,\cdots,0)-A_{nk}l_k\big) \\
& +(p_1,\cdots,p_{k-1},p_kA_{kk},p_{k+1},\cdots,p_n).
\end{aligned}
$$

Since $\boldsymbol{NR}(0,\cdots,0,l_k,0,\cdots,0)$ represents the set of all preferences $(x_1,\cdots,x_n)$ such that type j where $j\in\{1,2,\cdots,k-1,k+1,\cdots,n\}$ all deciding d_2 (x_j is being low that is why $p_j\in(-\infty,0]$) while type t_k preferences x_k is moving from $X_k(0,\cdots,0)$ towards $X_k(0,\cdots,0,m_k,0,\cdots,0)$ (that is why $p_k\in[0,1]$) and each move costs $-A_{kk}$, so total cost is $-A_{kk}l_k$. Hence, $\boldsymbol{NR}(0,\cdots,0,l_k,0,\cdots,0)$ can be finally written as

$$
\begin{aligned}
\boldsymbol{N}(0,0,\cdots,0,l_k,0\cdots,0) = & \{C+\vec{Z}(0,0,\cdots,0,l_k,0\cdots,0) \\
& +(p_1,p_2,\cdots,p_{k-1},p_kA_{kk},p_{k+1},\cdots,p_n): \\
& p_k\in[0,1],\ p_j\in(-\infty,0] \\
& \forall j\in\{1,2,\cdots,k-1,k+1,\cdots,n\}.
\end{aligned}
$$

□

The proof of Theorem 3.23 for the other cases follows similarly to the proof of the first case.

3.5 Mixed Nash equilibria

Recall that from Definition 3.2 we have

$$\boldsymbol{I}=\boldsymbol{I_1}\sqcup\boldsymbol{I_2}\sqcup\cdots\sqcup\boldsymbol{I_n},$$

where

$$\boldsymbol{I} = \{i = (i_1, i_2, \cdots, i_n) \in \mathbb{R}^n : i_k \in \boldsymbol{I_k},\ \ k \in \{1, \cdots, n\}\}.$$

Each player $i_k \in \boldsymbol{I_k}$ is assumed to make one decision $d \in \boldsymbol{D} = \{d_1, d_2\}$.

Given a mixed strategy $S = (S_1, S_2, \cdots, S_n)$, we describe the mixed decision of the players of type t_k by a mixed strategy map

$$S_k : \boldsymbol{I_k} \longrightarrow [\mathbf{0}, \mathbf{1}]$$

that associates to each player $i_k \in I_k$ the probability $p_{i_k} = S(i_k)$ to decide $d_1 \in \boldsymbol{D}$, where $k \in \{1, \cdots, n\}$. Hence, each player $i_k \in \boldsymbol{I_k}$ decides $d_2 \in \boldsymbol{D}$ with probability $1 - p_{i_k} = 1 - S(i_k)$. We assume that the decisions of the players are taken independently.

Define

$$P_k = \sum_{i=1}^{m_k} p_{i_k},$$

and

$$P_{i_k} = P_k - p_{i_k},$$

for $k = 1, 2, \cdots, n$.

For every player $i_k \in \boldsymbol{I_k}$, the d_1-fitness function for players of type t_k $f_{d_1,k} : [0, 1] \times [0, m_1] \times \cdots \times [0, m_n] \longrightarrow \mathbb{R}^+$ is defined by

$$\begin{aligned} f_{d_1,k}(p_{i_k}; P_1, P_2, \cdots, P_n) =& \omega_k^{d_1} + \alpha_{k1}^{d_1} P_1 + \cdots + \alpha_{kk-1}^{d_1} P_{k-1} + \alpha_{kk}^{d_1} P_{i_k} + \alpha_{kk+1}^{d_1} P_{k+1} \\ & + \cdots + \alpha_{kn}^{d_1} P_n; \end{aligned} \tag{3.28}$$

and the d_2-fitness function $f_{d_2,k} : [0, 1] \times [0, m_1] \times \cdots \times [0, m_n] \longrightarrow \mathbb{R}^+$ is defined by

$$\begin{aligned} f_{d_2,k}(p_{i_k}; P_1, P_2, \cdots, P_n) \;\; = \;\; & \omega_k^{d_2} + \alpha_{k1}^{d_2}(m_2 - P_1) + \cdots + \alpha_{kk-1}^{d_2}(m_{k-1} - P_{k-1}) \\ & + \alpha_{kk}^{d_2}(m_k - 1 - P_{i_k}) + \alpha_{kk+1}^{d_2}(m_{k+1} - P_{k+1}) \\ & + \cdots + \alpha_{kn}^{d_2}(m_n - P_n). \end{aligned} \tag{3.29}$$

Definition 3.24. *Let $S = (S_1, S_2, \cdots, S_n)$ be a mixed strategy where*

$$S_k : \boldsymbol{I_k} \longrightarrow [\mathbf{0}, \mathbf{1}], \text{ where } k \in \{1, 2, \cdots, n\}.$$

For every individual $i_k \in \boldsymbol{I_k}$, The utility function

$$\Pi_k : [0,1] \times [0, m_1] \times \cdots \times [0, m_n] \longrightarrow \mathbb{R}^+,$$

is defined by

$$\Pi_k(p_{i_k}; P_1, \cdots, P_n) = p_{i_k} f_{d_1,k}(p_{i_k}; P_1, \cdots, P_n) + (1 - p_{i_k}) f_{d_2,k}(p_{i_k}; P_1, \cdots, P_n).$$

Lemma 3.25. *Let $S_k : \boldsymbol{I_k} \longrightarrow [\mathbf{0}, \mathbf{1}]$ be a mixed Nash equilibrium for all $k = 1, 2, \cdots, n$. If $0 < p_{i_k} < 1$, then*

$$\begin{aligned} x_k &= -A_{kk}(P_k - p_{i_k}) - A_{k1}P_1 - A_{k2}P_2 - \cdots - A_{kk-1}P_{k-1} - A_{kk+1}P_{k+1} - \cdots \\ &\quad -A_{kn}P_n + X_k(d_2, d_2, \cdots, d_2). \end{aligned} \tag{3.30}$$

Proof. Let $S_k : \boldsymbol{I_k} \longrightarrow [\mathbf{0}, \mathbf{1}]$ be a mixed Nash equilibrium for all $k = 1, 2, \cdots, n$. For every $p \in [0, 1]$, we have

$$\Pi_k(p_{i_k}; P_1, P_2, \cdots, P_n) \geq \Pi_k(p; P_1, P_2, \cdots, P_{k-1}, P_k - p_{i_k} + p, P_{k+1}, \cdots, P_n).$$

If $0 < p_{i_k} < 1$, we get

$$f_{d_1,k}(p_{i_k}, P_1, P_2, \cdots, P_n) = f_{d_2,k}(p_{i_k}, P1, P_2, \cdots, P_n).$$

Substitute the fitness functions we get

$$\begin{aligned} &\omega_k^{d_1} + \alpha_{k1}^{d_1} P_1 + \cdots + \alpha_{kk-1}^{d_1} P_{k-1} + \alpha_{kk}^{d_1} P_{i_k} + \alpha_{kk+1}^{d_1} P_{k+1} + \cdots + \alpha_{kn}^{d_1} P_n = \\ &\omega_k^{d_2} + \alpha_{k1}^{d_2}(m_2 - P_1) + \cdots + \alpha_{kk-1}^{d_2}(m_{k-1} - P_{k-1}) + \alpha_{kk}^{d_2}(m_k - 1 - P_{i_k}) \\ &+\alpha_{kk+1}^{d_2}(m_{k+1} - P_{k+1}) + \cdots + \alpha_{kn}^{d_2}(m_n - P_n). \end{aligned} \tag{3.31}$$

Rearranging the previous equation we obtain

$$\begin{aligned}
&\omega_k^{d_1} - \omega_k^{d_2} = -\alpha_{k1}^{d_1} P_1 - \cdots - \alpha_{kk-1}^{d_1} P_{k-1} - \alpha_{kk}^{d_1} P_{i_k} - \alpha_{kk+1}^{d_1} P_{k+1} - \cdots - \alpha_{kn}^{d_1} P_n \\
&+\alpha_{k1}^{d_2}(m_2 - P_1) + \cdots + \alpha_{kk-1}^{d_2}(m_{k-1} - P_{k-1}) + \alpha_{kk}^{d_2}(m_k - 1 - P_{i_k}) \qquad (3.32) \\
&+\alpha_{kk+1}^{d_2}(m_{k+1} - P_{k+1}) + \cdots + \alpha_{kn}^{d_2}(m_n - P_n).
\end{aligned}$$

Using the Definition 3.5 and expand the brackets, we rearrange 3.32 to get

$$\begin{aligned}
&x_k = -\alpha_{k1}^{d_1} P_1 - \cdots - \alpha_{kk-1}^{d_1} P_{k-1} - \alpha_{kk}^{d_1} P_{i_k} - \alpha_{kk+1}^{d_1} P_{k+1} - \cdots - \alpha_{kn}^{d_1} P_n \\
&+\alpha_{k1}^{d_2} m_2 - \alpha_{k1}^{d_2} P_1 + \cdots + \alpha_{kk-1}^{d_2} m_{k-1} - \alpha_{kk-1}^{d_2} P_{k-1} + \alpha_{kk}^{d_2} m_k - \alpha_{kk}^{d_2} - \alpha_{kk}^{d_2} P_{i_k} \\
&+\alpha_{kk+1}^{d_2} m_{k+1} - \alpha_{kk+1}^{d_2} P_{k+1} + \cdots + \alpha_{kn}^{d_2} m_n - \alpha_{kn}^{d_2} P_n.
\end{aligned}$$

Take common factor from previous equalty to get

$$\begin{aligned}
&x_k = -(\alpha_{k1}^{d_1} + \alpha_{k1}^{d_2}) P_1 - \cdots - (\alpha_{kk-1}^{d_1} + \alpha_{kk-1}^{d_2}) P_{k-1} - (\alpha_{kk}^{d_1} + \alpha_{kk}^{d_2}) P_{i_k} \\
&-(\alpha_{kk+1}^{d_1} + \alpha_{kk+1}^{d_2}) P_{k+1} - \cdots - (\alpha_{kn}^{d_1} + \alpha_{kn}^{d_2}) P_n + \alpha_{k1}^{d_2} m_2 + \cdots + \alpha_{kk-1}^{d_2} m_{k-1} \\
&+\alpha_{kk}^{d_2} m_k - \alpha_{kk}^{d_2} + \alpha_{kk+1}^{d_2} m_{k+1} + \cdots + \alpha_{kn}^{d_2} m_n.
\end{aligned}$$

Using the definition of the coordinates of the influence crowding matrix, we rearrange the previous equality to get

$$\begin{aligned}
&x_k = -A_{k1} P_1 - \cdots - A_{kk-1} P_{k-1} - A_{kk} P_{i_k} - A_{kk+1} P_{k+1} - \cdots - A_{kn} P_n \\
&+\alpha_{k1}^{d_2} m_2 + \cdots + \alpha_{kk-1}^{d_2} m_{k-1} + \alpha_{kk}^{d_2}(m_k - 1) + \alpha_{kk+1}^{d_2} m_{k+1} + \cdots + \alpha_{kn}^{d_2} m_n.
\end{aligned}$$

Which can simplified using threshold $X_k(0, 0, \cdots, 0)$ from Theorem 3.12 to

$$x_k = -A_{k1} P_1 - \cdots - A_{kk-1} P_{k-1} - A_{kk} P_{i_k} - A_{kk+1} P_{k+1} - \cdots - A_{kn} P_n + X_k(0, 0, \cdots, 0).$$

□

Chapter 4

Applications in Economics

In this chapter, we will introduce a special case of our decision model by including three types of tourists, who will distribute among two resorts: Beach resort and Mountain resort.

4.1 General resort pricing model

In this section, we will explain the parameters of our model for resort interest. We will assume that $\boldsymbol{T}$ is the set of n types of homogeneous tourists, where each t_k type have m_k tourist. Each tourists have to choose between spending their holidays in a Beach resort $B = d_1$ or in a Mountain resort $M = d_2$, i.e. $d \in D = \{d_1, d_2\} = \{B, M\}$.
Let P be the price vector whose coordinates $p^r > 0$ indicates the standard price of the resort $r \in \{B, M\}$ for each resort, independently of its type. That is,

$$P = (p^B, p^M).$$

Let $\boldsymbol{L}$ be preference location matrix that reflects the tourists taste type and given by

$$\boldsymbol{L} = \begin{pmatrix} \omega_1^B & \omega_1^M \\ \omega_2^B & \omega_2^M \\ \omega_3^B & \omega_3^M \\ \vdots & \vdots \\ \omega_n^B & \omega_n^M \end{pmatrix} \in \mathbb{R}^{n\times 2}.$$

Let N_r be the interacting matrix that reflects the crowding effect of resort $r \in D$ and given by

$$\boldsymbol{N_r} = \begin{pmatrix} \alpha_{11}^r & \alpha_{12}^r & \alpha_{13}^r & \dots & \alpha_{1n}^r \\ \alpha_{21}^r & \alpha_{22}^r & \alpha_{23}^r & \dots & \alpha_{2n}^r \\ \alpha_{31}^r & \alpha_{32}^r & \alpha_{33}^r & \dots & \alpha_{3n}^r \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{n1}^r & \alpha_{n2}^r & \alpha_{n3}^r & \dots & \alpha_{nn}^r \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

Let $\mathcal{A}$ be the partial crowding matrix whose coordinates given by

$$\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1n} \\ A_{21} & A_{22} & A_{23} & \dots & A_{2n} \\ A_{31} & A_{32} & A_{33} & \dots & A_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & A_{n3} & \dots & A_{nn} \end{pmatrix}.$$

where

$$A_{kj} = \alpha_{kj}^B + \alpha_{kj}^M,$$

for all $k, j \in \{1, 2, 3, \cdots, n\}$. The partial coordinates encode all relevant information for the existence of Nash equilibria strategies.

We now define the corresponding utility functions $\Pi_k : \boldsymbol{D} \times \boldsymbol{O} \longrightarrow \mathbb{R}$ is the utility function of the tourist with type t_k who choose resort $r \in \boldsymbol{D}$ is defined by

$$\begin{aligned}
\Pi_k(B; l_1, \cdots, l_k, \cdots, l_n) &= -p^B + \omega_k^B + \alpha_{k1}^B l_1 + \alpha_{k2}^B l_2 + \cdots + \alpha_{kk}^B (l_k - 1) + \cdots \\
&\quad + \alpha_{kn}^B l_n + \epsilon_k(B) \qquad (4.1) \\
\Pi_k(M; l_1, \cdots, l_k, \cdots, l_n) &= -p^M + \omega_k^M + \alpha_{k1}^M (m_1 - l_1) + \alpha_{k2}^B (m_2 - l_2) + \cdots + \\
&\quad \alpha_{kk}^M (m_k - l_k - 1) + \cdots + \alpha_{kn}^M (m_n - l_n) \qquad (4.2) \\
&\quad + \epsilon_k(M),
\end{aligned}$$

where $k \in \{1, 2, \cdots, n\}$.

4.2 Particular case for three types of tourists

We consider three types of tourists. Hence, the utility functions in (3.5) and (3.6) are simplified as follows:

$\Pi_1 : \boldsymbol{r} \times \boldsymbol{O} \longrightarrow \mathbb{R}$ is the utility function of the tourist with type t_1 who choose resort $r \in \boldsymbol{D}$ is defined by

$$\begin{aligned}
\Pi_1(B; l_1, l_2, l_3) &= -p^B + \omega_1^B + \alpha_{11}^B (l_1 - 1) + \alpha_{12}^B l_2 + \alpha_{13}^B l_3 + \epsilon_1(B) \qquad (4.3) \\
\Pi_1(M; l_1, l_2, l_3) &= -p^M + \omega_1^M + \alpha_{11}^M (m_1 - l_1 - 1) + \alpha_{12}^B (m_2 - l_2) \\
&\quad + \alpha_{13}^M (m_3 - l_3) + \epsilon_1(M) \qquad (4.4)
\end{aligned}$$

$\Pi_2 : \boldsymbol{r} \times \boldsymbol{O} \longrightarrow \mathbb{R}$ is the utility function of the tourist with type t_2 who choose resort $r \in \boldsymbol{D}$ is defined by

$$\begin{aligned}
\Pi_2(B; l_1, l_2, l_3) &= -p^B + \omega_2^B + \alpha_{21}^B l_1 + \alpha_{22}^B (l_2 - 1) + \alpha_{23}^B l_3 + \epsilon_2(B) \qquad (4.5) \\
\Pi_2(M; l_1, l_2, l_3) &= -p^M + \omega_2^M + \alpha_{21}^M (m_1 - l_1) + \alpha_{22}^B (m_2 - l_2 - 1) \\
&\quad + \alpha_{23}^M (m_3 - l_3) + \epsilon_2(M), \qquad (4.6)
\end{aligned}$$

and $\Pi_3 : \boldsymbol{r} \times \boldsymbol{O} \longrightarrow \mathbb{R}$ is the utility function of the tourist with type t_3 who choose resort $r \in \boldsymbol{D}$ is defined by

$$\Pi_3(B; l_1, l_2, l_3) = -p^B + \omega_3^B + \alpha_{31}^B l_1 + \alpha_{32}^B l_2 + \alpha_{33}^B (l_3 - 1) + \epsilon_3(B) \quad (4.7)$$

$$\Pi_3(M; l_1, l_2, l_3) = -p^M + \omega_3^M + \alpha_{31}^M (m_1 - l_1) + \alpha_{32}^B (m_2 - l_2) + \alpha_{33}^M (m_3 - l_3 - 1) + \epsilon_3(M). \quad (4.8)$$

Definition 4.1. *We define the following location preferences*

$x = \omega_1^B - \omega_1^M$ *is the horizontal relative location preference of the tourists with type* t_1,

$y = \omega_2^B - \omega_2^M$ *is the vertical relative location preference of the tourists with type* t_2 *and*

$z = \omega_3^B - \omega_3^M$ *is the depth relative location preference of the tourists with type* t_3.

Let (x, y, z) be relative location preference of tourists with types (t_1, t_2, t_3) respectively.

Definition 4.2. *Let* $p = p^B - p^M$ *be the significant difference price which takes real values.*

That is, if $p = 0$ then $p^B = p^M$, if $p > 0$ then $p^B > p^M$ and if $p < 0$ then $p^B < p^M$.

Definition 4.3. *A united strategy* (R_1, R_2, R_3) *is the set of all relative prices p for which the strategy* (R_1, R_2, R_3) *is a Nash equilibrium, where* $R_k \in \{B, M\}$ *for all* $k = 1, 2, 3$.

Given a triple (x, y, z) of relative location preferences, the Nash equilibrium prices interval $\boldsymbol{NP}(R_1, R_2, R_3)$ of a united strategy (R_1, R_2, R_3) is the set of all relative prices p for which the strategy (R_1, R_2, R_3) is a Nash equilibrium. So, our aim in next section is to determine and characterize all Nash equilibrium prices intervals.

4.3 Nash Equilibrium prices

In this section, we will determine and characterize all Nash equilibrium prices intervals in tourist model for a given preferences (x, y, z). As we proved in Lemma 3.9 there are 2^n united strategies, where n is the number of homogeneous types. So, as we have here three types of homogeneous tourists, so we can observe that there are eight (2^3) distinct united strategies as follow:

1. (B,B,B) strategy - all tourists choose the resort B;
2. (B,B,M) strategy - all tourists with type t_1 and t_2 choose the resort B, but all tourists with type t_3 choose the resort M;
3. (B,M,B) strategy - all tourists with type t_1 and t_3 choose the resort B, but all tourists with type t_2 choose the resort M;
4. (B,M,M) strategy - all tourists with type t_1 choose the resort B, but all tourists with type t_2 and t_3 choose the resort M;
5. (M,B,B) strategy - all tourist with type t_2 and t_3 choose the resort B, but all tourists with type t_1 choose the resort M;
6. (M,B,M) strategy - all tourists with type t_2 choose the resort B, but all tourists, with type t_1 and t_3 choose the resort M;
7. (M,M,B) strategy - all tourists with type t_3 choose the resort B, but all tourists with type t_1 and t_2 choose the resort M;
8. (M,M,M) strategy - all tourists choose the resort M.

Theorem 4.4. *Given the preferences x, y, z for three types.*
The Nash Equilibrium prices interval $\boldsymbol{NP}(B, B, B)$ for which (B, B, B) is Nash Equilibrium is the set

$$\begin{aligned} \boldsymbol{NP}(B,B,B) = \{p \in \mathbb{R} : p &\leq x - H(B,B,B) \textit{ and} \\ p &\leq y - V(B,B,B) \textit{ and} \\ p &\leq z - D(B,B,B)\}, \end{aligned} \tag{4.9}$$

where the horizontal $H(B,B,B)$, *vertical* $V(B,B,B)$ *and depth* $D(B,B,B)$ *strategic thresholds of the* (B,B,B) *strategy are respectively, given by*

$$\begin{aligned}
H(B,B,B) &= -\alpha_{11}^{B}(m_1-1)-\alpha_{12}^{B}m_2-\alpha_{13}^{B}m_3+\epsilon_1(M)-\epsilon_1(B),\\
V(B,B,B) &= -\alpha_{21}^{B}m_1-\alpha_{22}^{B}(m_2-1)-\alpha_{23}^{B}m_3+\epsilon_2(M)-\epsilon_2(B),\\
D(B,B,B) &= -\alpha_{31}^{B}m_1-\alpha_{32}^{B}m_2-\alpha_{33}^{B}(m_3-1)+\epsilon_3(M)-\epsilon_3(B).
\end{aligned}$$

The Nash equilibrium prices interval $\boldsymbol{NP}(B,B,M)$ *for which* (B,B,M) *is Nash Equilibrium is the set*

$$\begin{aligned}
\boldsymbol{NP}(B,B,M)=\{p\in\mathbb{R}: p &\le x-H(B,B,M) \text{ and}\\
p &\le y-V(B,B,M) \text{ and}\\
p &\ge z-D(B,B,M)\},
\end{aligned}$$

where the horizontal $H(B,B,M)$, *vertical* $V(B,B,M)$ *and depth* $D(B,B,M)$ *strategic thresholds of the* (B,B,M) *strategy are respectively, given by*

$$\begin{aligned}
H(B,B,M) &= -\alpha_{11}^{B}(m_1-1)-\alpha_{12}^{B}m_2+\alpha_{13}^{M}m_3+\epsilon_1(M)-\epsilon_1(B),\\
V(B,B,M) &= -\alpha_{21}^{B}m_1-\alpha_{22}^{B}(m_2-1)+\alpha_{23}^{M}m_3+\epsilon_2(M)-\epsilon_2(B),\\
D(B,B,M) &= -\alpha_{31}^{B}m_1-\alpha_{32}^{B}m_2+\alpha_{33}^{M}(m_3-1)+\epsilon_3(M)-\epsilon_3(B).
\end{aligned}$$

The Nash equilibrium prices interval $\boldsymbol{NP}(B,M,B)$ *for which* (B,M,B) *is Nash Equilibrium is the set*

$$\begin{aligned}
\boldsymbol{NP}(B,M,B)=\{p\in\mathbb{R}: p &\le x-H(B,M,B) \text{ and}\\
p &\ge y-V(B,M,B) \text{ and}\\
p &\le z-D(B,M,B)\},
\end{aligned}$$

where the horizontal $H(B,M,B)$, *vertical* $V(B,M,B)$ *and depth* $D(B,M,B)$ *strategic thresholds of the* (B,M,B) *strategy are respectively, given by*

$$\begin{aligned}
H(B,M,B) &= -\alpha_{11}^{B}(m_1-1)+\alpha_{12}^{M}m_2-\alpha_{13}^{B}m_3+\epsilon_1(M)-\epsilon_1(B),\\
V(B,M,B) &= -\alpha_{21}^{B}m_1+\alpha_{22}^{M}(m_2-1)-\alpha_{23}^{B}m_3+\epsilon_2(M)-\epsilon_2(B),
\end{aligned}$$

$$D(B,M,B) = -\alpha_{31}^{B} m_1 + \alpha_{32}^{M} m_2 - \alpha_{33}^{B}(m_3 - 1) + \epsilon_3(M) - \epsilon_3(B).$$

The Nash equilibrium prices interval $\boldsymbol{NP}(B,M,M)$ *for which* (B,M,M) *is Nash Equilibrium is the set*

$$\begin{aligned} \boldsymbol{NP}(B,M,M) = \{p \in \mathbb{R} : p &\leq x - H(B,M,M) \text{ and} \\ p &\geq y - V(B,M,M) \text{ and} \\ p &\geq z - D(B,M,M)\}, \end{aligned}$$

where the horizontal $H(B,M,M)$*, vertical* $V(B,M,M)$ *and depth* $D(B,M,M)$ *strategic thresholds of the* (B,M,M) *strategy are respectively, given by*

$$\begin{aligned} H(B,M,M) &= -\alpha_{11}^{B}(m_1 - 1) + \alpha_{12}^{M} m_2 + \alpha_{13}^{M} m_3 + \epsilon_1(M) - \epsilon_1(B), \\ V(B,M,M) &= -\alpha_{21}^{B} m_1 + \alpha_{22}^{M}(m_2 - 1) + \alpha_{23}^{M} m_3 + \epsilon_2(M) - \epsilon_2(B), \\ D(B,M,M) &= -\alpha_{31}^{B} m_1 + \alpha_{32}^{M} m_2 + \alpha_{33}^{M}(m_3 - 1) + \epsilon_3(M) - \epsilon_3(B). \end{aligned}$$

The Nash equilibrium prices interval $\boldsymbol{NP}(M,B,B)$ *for which* (M,B,B) *is Nash Equilibrium is the set*

$$\begin{aligned} \boldsymbol{NP}(M,B,B) = \{p \in \mathbb{R} : p &\geq x - H(M,B,B) \text{ and} \\ p &\leq y - V(M,B,B) \text{ and} \\ p &\leq z - D(M,B,B)\}, \end{aligned}$$

where the horizontal $H(M,B,B)$*, vertical* $V(M,B,B)$ *and depth* $D(M,B,B)$ *strategic thresholds of the* (M,B,B) *strategy are respectively, given by*

$$\begin{aligned} H(M,B,B) &= \alpha_{11}^{M}(m_1 - 1) - \alpha_{12}^{B} m_2 - \alpha_{13}^{B} m_3 + \epsilon_1(M) - \epsilon_1(B), \\ V(M,B,B) &= \alpha_{21}^{M} m_1 - \alpha_{22}^{B}(m_2 - 1) - \alpha_{23}^{B} m_3 + \epsilon_2(M) - \epsilon_2(B), \\ D(M,B,B) &= \alpha_{31}^{M} m_1 - \alpha_{32}^{B} m_2 - \alpha_{33}^{B}(m_3 - 1) + \epsilon_3(M) - \epsilon_3(B). \end{aligned}$$

The Nash equilibrium prices interval $\boldsymbol{NP}(M,B,M)$ *for which* (M,B,M) *is Nash Equilibrium is the set*

$$\begin{aligned}\boldsymbol{NP}(M,B,M)=\{p\in\mathbb{R}:p &\geq x-H(M,B,M)\ \textit{and}\\ p &\leq y-V(M,B,M)\ \textit{and}\\ p &\geq z-D(M,B,M)\},\end{aligned}$$

where the horizontal $H(M,B,M)$*, vertical* $V(M,B,M)$ *and depth* $D(M,B,M)$ *strategic thresholds of the* (M,B,M) *strategy are respectively, given by*

$$\begin{aligned}H(M,B,M)&=\alpha_{11}^{M}(m_1-1)-\alpha_{12}^{B}m_2+\alpha_{13}^{M}m_3+\epsilon_1(M)-\epsilon_1(B),\\ V(M,B,M)&=\alpha_{21}^{M}m_1-\alpha_{22}^{B}(m_2-1)+\alpha_{23}^{M}m_3+\epsilon_2(M)-\epsilon_2(B),\\ D(M,B,M)&=\alpha_{31}^{M}m_1-\alpha_{32}^{B}m_2+\alpha_{33}^{M}(m_3-1)+\epsilon_3(M)-\epsilon_3(B).\end{aligned}$$

The Nash equilibrium prices interval $\boldsymbol{NP}(M,M,B)$ *for which* (M,M,B) *is Nash Equilibrium is the set*

$$\begin{aligned}\boldsymbol{NP}(M,M,B)=\{p\in\mathbb{R}:p &\geq x-H(M,M,B)\ \textit{and}\\ p &\geq y-V(M,M,B)\ \textit{and}\\ p &\leq z-D(M,M,B)\},\end{aligned}$$

where the horizontal $H(M,M,B)$*, vertical* $V(M,M,B)$ *and depth* $D(M,M,B)$ *strategic thresholds of the* (M,M,B) *strategy are respectively, given by*

$$\begin{aligned}H(M,M,B)&=\alpha_{11}^{M}(m_1-1)+\alpha_{12}^{M}m_2-\alpha_{13}^{B}m_3+\epsilon_1(M)-\epsilon_1(B),\\ V(M,M,B)&=\alpha_{21}^{M}m_1+\alpha_{22}^{M}(m_2-1)-\alpha_{23}^{B}m_3+\epsilon_2(M)-\epsilon_2(B),\\ D(M,M,B)&=\alpha_{31}^{M}m_1+\alpha_{32}^{M}m_2-\alpha_{33}^{B}(m_3-1)+\epsilon_3(M)-\epsilon_3(B).\end{aligned}$$

The Nash equilibrium prices interval $\boldsymbol{NP}(M,M,M)$ *for which* (M,M,M) *is Nash Equilibrium is the set*

$$\begin{aligned}\boldsymbol{NP}(M,M,M)=\{p\in\mathbb{R}:p &\geq x-H(M,M,M)\ \textit{and}\\ p &\geq y-V(M,M,M)\ \textit{and}\\ p &\geq z-D(M,M,M)\},\end{aligned}\tag{4.10}$$

where the horizontal $H(M,M,M)$*, vertical* $V(M,M,M)$ *and depth* $D(M,M,M)$ *strategic thresholds of the* (M,M,M) *strategy are respectively, given by*

$$\begin{aligned}H(M,M,M)&=\alpha_{11}^{M}(m_1-1)+\alpha_{12}^{M}m_2+\alpha_{13}^{M}m_3+\epsilon_1(M)-\epsilon_1(B),\\ V(M,M,M)&=\alpha_{21}^{M}m_1+\alpha_{22}^{M}(m_2-1)+\alpha_{23}^{M}m_3+\epsilon_2(M)-\epsilon_2(B),\\ D(M,M,M)&=\alpha_{31}^{M}m_1+\alpha_{32}^{M}m_2+\alpha_{33}^{M}(m_3-1)+\epsilon_3(M)-\epsilon_3(B).\end{aligned}$$

Proof. The united strategy (B,B,B) is NE iff:

$$\begin{aligned}\Pi_1(B;m_1,m_2,m_3) &\geq \Pi_1(M;m_1-1,m_2,m_3)\ \textit{and}\\ \Pi_2(B;m_1,m_2,m_3) &\geq \Pi_2(M;m_1,m_2-1,m_3)\ \textit{and}\\ \Pi_3(B;m_1,m_2,m_3) &\geq \Pi_3(M;m_1,m_2,m_3-1).\end{aligned}\tag{4.11}$$

Substituting the utility functions given in (4.3) and (4.4) in inequality (4.11) we get

$$\begin{aligned}&-p^B+\omega_1^B+\alpha_{11}^B(m_1-1)+\alpha_{12}^Bm_2+\alpha_{13}^Bm_3+\epsilon_1(B)\geq -p^M+\omega_1^M\\ &+\alpha_{11}^M(m_1-(m_1-1)-1)+\alpha_{12}^M(m_2-m_2)+\alpha_{13}^M(m_3-m_3)+\epsilon_1(M)\quad \textit{and}\\ &-p^B+\omega_2^B+\alpha_{21}^Bm_1+\alpha_{22}^B(m_2-1)+\alpha_{23}^Bm_3+\epsilon_2(B)\geq -p^M+\omega_2^M\\ &+\alpha_{21}^M(m_1-m_1)+\alpha_{22}^M(m_2-(m_2-1)-1)+\alpha_{23}^M(m_3-m_3)+\epsilon_2(M)\quad \textit{and}\\ &-p^B+\omega_3^B+\alpha_{n1}^Bm_1+\alpha_{32}^Bm_2+\alpha_{33}^B(m_3-1)+\epsilon_n(B)\geq -p^M+\omega_3^M\\ &+\alpha_{n1}^M(m_1-m_1)+\alpha_{32}^M(m_2-m_2)+\alpha_{33}^M(m_3-(m_3-1)-1)+\epsilon_n(M).\end{aligned}\tag{4.12}$$

Rearrange the previous inequalities (4.12), we obtain

$$
\begin{aligned}
\omega_1^B - \omega_1^M &\geq p^B - p^M - \alpha_{11}^B(m_1 - 1) - \alpha_{12}^B m_2 - \alpha_{13}^B m_3 + \epsilon_1(M) - \epsilon_1(B) \\
\omega_2^B - \omega_2^M &\geq p^B - p^M - \alpha_{21}^B m_1 - \alpha_{22}^B(m_2 - 1) - \alpha_{23}^B m_3 + \epsilon_2(M) - \epsilon_2(B) \\
\omega_3^B - \omega_3^M &\geq p^B - p^M - \alpha_{n1}^B m_1 - \alpha_{32}^B m_2 - \alpha_{33}^B(m_3 - 1) + \epsilon_3(M) - \epsilon_3(B).
\end{aligned}
$$

Substituting the values of the relative decisions from Definitions 4.1 and 4.2 together with the strategic thresholds $H(B,B,B)$, $V(B,B,B)$ and $D(B,B,B)$ given in (4.15), the last inequalities simplify to

$$
\begin{aligned}
x &\geq p + H(B,B,B) \quad \textit{and} \\
y &\geq p + V(B,B,B) \quad \textit{and} \qquad (4.13) \\
z &\geq p + D(B,B,B).
\end{aligned}
$$

Rearrange inequality in (4.13) we obtain

$$
\begin{aligned}
p &\leq x - H(B,B,B) \quad \textit{and} \\
p &\leq y - V(B,B,B) \quad \textit{and} \\
p &\leq z - D(B,B,B).
\end{aligned}
$$

That is, since x, y, z are given, it follows that the Nash domain prices for which (B,B,B) is NE is the interval

$$
\begin{aligned}
\boldsymbol{NP}(B,B,B) = \{p \in \mathbb{R} : p \leq min\, \{x &- H(B,B,B), \\
y &- V(B,B,B), \\
z &- D(B,B,B)\}\}.
\end{aligned}
$$

The proof of the other cases follow similarly to the proof of the first case. □

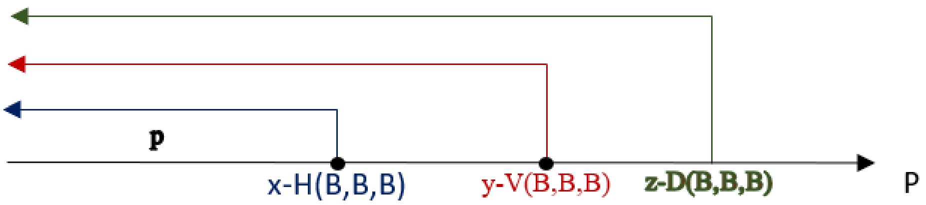


FIGURE 4.1: The Nash equilibrium prices interval $\boldsymbol{NP}(B,B,B)$.

4.4 Monopoly versus duopoly

As a result of Theorem 4.4 first case, if $p \in \boldsymbol{NP}(B,B,B)$ then all tourist will enjoy there time only in resort B and no one go to resorts M. Hence, there will be a monopoly position for resort B while resort M goes to bankruptcy.

Remark 4.5. *Based on Theorem 4.4 first case, we can represent the Nash Equilibrium price interval as we shown in the following Figure 4.1, which is the set*

$$\begin{aligned} \boldsymbol{NP}(B,B,B) = \{p \in \mathbb{R} : p &\leq x - H(B,B,B) \ \textit{and} \\ p &\leq y - V(B,B,B) \ \textit{and} \\ p &\leq z - D(B,B,B)\}. \end{aligned} \tag{4.14}$$

Remark 4.6. *As a result of Theorem 4.4, we have two monopoly cases, one for Beach resort and the second for Mountain resort, and this happened in the following cases.*

First, when (B,B,B) is Nash Equilibrium, then the (4.9) set can written as

$$\begin{aligned} \boldsymbol{NP}(B,B,B) = \{p \in \mathbb{R} : p \leq min\ \{x &- H(B,B,B), \\ y &- V(B,B,B), \\ z &- D(B,B,B)\}, \end{aligned}$$

where the monopoly here for Resort B.

Second, when (M, M, M) is Nash Equilibrium, then the (4.10) set can written as

$$\boldsymbol{NP}(M,M,M) = \{p \in \mathbb{R} : p \geq max\ \{x - H(M,M,M), \\ y - V(M,M,M), \\ z - D(M,M,M)\},$$

as a result, if $p \in \boldsymbol{NP}(M,M,M)$, then all tourists will enjoy there Holiday time only in resort M and no one go to resort B. Hence, there will be a monopoly position for Resort M while resort B goes to bankruptcy.

The other six cases given in Theorem (4.4) are representing duopoly between resort B and resort M, where given the relative price p, both resorts are competitive in the market.

4.5 Alternative resort model

In this section we can have similar results as in section 4.3 but for given relative price p. See next remark

Remark 4.7. *Note that we can derived similar result to Theorem 4.4 using the following assumption:*

Given the relative price p, then the Nash Equilibrium region is given by

$$\boldsymbol{NR}(B,B,B) = \{(x,y,z) \in \mathbb{R}^3 \ : \ x \geq p + H(B,B,B), \\ y \geq p + V(B,B,B), \\ z \geq p + Z(B,B,B)\}.$$

We represent this case in Figure 4.2.

So, for a given relative price $p \in \mathbb{R}$, and for every $(x, y, z) \in \boldsymbol{NR}(B, B, B)$ we are sure that all tourists go to Beach resort, and mountain resort will goes to bankruptcy.

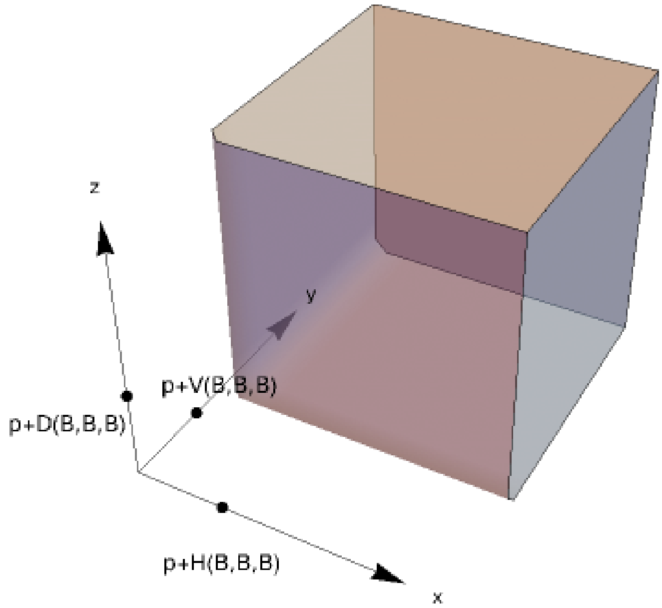


FIGURE 4.2: The $\boldsymbol{NR}(B, B, B)$ region for given relative price p.

Theorem 4.8. *Given the relative price p,*

then the Nash Equilibrium region $\boldsymbol{NR}(B, B, B)$ for (B, B, B) strategy is given by

$$\begin{aligned} \boldsymbol{NR}(B,B,B) = \{(x,y,z) \in \mathbb{R}^3 : x &\geq p + H(B,B,B) \ \textit{and} \\ y &\geq p + V(B,B,B) \ \textit{and} \\ z &\geq p + D(B,B,B)\}, \end{aligned}$$

where the horizontal $H(B, B, B)$, vertical $V(B, B, B)$ and depth $D(B, B, B)$ strategic thresholds of the (B, B, B) strategy are respectively, given by

$$\begin{aligned} H(B,B,B) &= -\alpha_{11}^B(m_1-1) - \alpha_{12}^B m_2 - \alpha_{13}^B m_3 + \epsilon_1(M) - \epsilon_1(B), \\ V(B,B,B) &= -\alpha_{21}^B m_1 - \alpha_{22}^B(m_2-1) - \alpha_{23}^B m_3 + \epsilon_2(M) - \epsilon_2(B), \quad (4.15) \\ D(B,B,B) &= -\alpha_{31}^B m_1 - \alpha_{32}^B m_2 - \alpha_{33}^B(m_3-1) + \epsilon_3(M) - \epsilon_3(B). \end{aligned}$$

The Nash equilibrium region $\boldsymbol{NR}(B, B, M)$ for (B, B, M) strategy is given by

$$\begin{aligned} \boldsymbol{NR}(B,B,M) = \{(x,y,z) \in \mathbb{R}^3 : x &\geq p + H(B,B,M) \ \textit{and} \\ y &\geq p + V(B,B,M) \ \textit{and} \\ z &\leq p + D(B,B,M)\}, \end{aligned}$$

where the horizontal $H(B,B,M)$*, vertical* $V(B,B,M)$ *and depth* $D(B,B,M)$ *strategic thresholds of the* (B,B,M) *strategy are respectively, given by*

$$\begin{aligned}
H(B,B,M) &= -\alpha_{11}^{B}(m_1-1) - \alpha_{12}^{B}m_2 + \alpha_{13}^{M}m_3 + \epsilon_1(M) - \epsilon_1(B),\\
V(B,B,M) &= -\alpha_{21}^{B}m_1 - \alpha_{22}^{B}(m_2-1) + \alpha_{23}^{M}m_3 + \epsilon_2(M) - \epsilon_2(B),\\
D(B,B,M) &= -\alpha_{31}^{B}m_1 - \alpha_{32}^{B}m_2 + \alpha_{33}^{M}(m_3-1) + \epsilon_3(M) - \epsilon_3(B).
\end{aligned}$$

The Nash equilibrium region $\boldsymbol{NR}(B,M,B)$ *for* (B,M,B) *strategy is given by*

$$\begin{aligned}
\boldsymbol{NR}(B,M,B) = \{(x,y,z) \in \mathbb{R}^3 : x &\geq p + H(B,M,B) \text{ and}\\
y &\leq p + V(B,M,B) \text{ and}\\
z &\geq p + D(B,M,B)\},
\end{aligned}$$

where the horizontal $H(B,M,B)$*, vertical* $V(B,M,B)$ *and depth* $D(B,M,B)$ *strategic thresholds of the* (B,M,B) *strategy are respectively, given by*

$$\begin{aligned}
H(B,M,B) &= -\alpha_{11}^{B}(m_1-1) + \alpha_{12}^{M}m_2 - \alpha_{13}^{B}m_3 + \epsilon_1(M) - \epsilon_1(B),\\
V(B,M,B) &= -\alpha_{21}^{B}m_1 + \alpha_{22}^{M}(m_2-1) - \alpha_{23}^{B}m_3 + \epsilon_2(M) - \epsilon_2(B),\\
D(B,M,B) &= -\alpha_{31}^{B}m_1 + \alpha_{32}^{M}m_2 - \alpha_{33}^{B}(m_3-1) + \epsilon_3(M) - \epsilon_3(B).
\end{aligned}$$

The Nash equilibrium region $\boldsymbol{NR}(B,M,M)$ *for* (B,M,M) *strategy is given by*

$$\begin{aligned}
\boldsymbol{NR}(B,M,M) = \{(x,y,z) \in \mathbb{R}^3 : x &\geq p + H(B,M,M) \text{ and}\\
y &\leq p + V(B,M,M) \text{ and}\\
z &\leq p + D(B,M,M)\},
\end{aligned}$$

where the horizontal $H(B,M,M)$*, vertical* $V(B,M,M)$ *and depth* $D(B,M,M)$ *strategic thresholds of the* (B,M,M) *strategy are respectively, given by*

$$\begin{aligned}
H(B,M,M) &= -\alpha_{11}^{B}(m_1-1) + \alpha_{12}^{M}m_2 + \alpha_{13}^{M}m_3 + \epsilon_1(M) - \epsilon_1(B),\\
V(B,M,M) &= -\alpha_{21}^{B}m_1 + \alpha_{22}^{M}(m_2-1) + \alpha_{23}^{M}m_3 + \epsilon_2(M) - \epsilon_2(B),\\
D(B,M,M) &= -\alpha_{31}^{B}m_1 + \alpha_{32}^{M}m_2 + \alpha_{33}^{M}(m_3-1) + \epsilon_3(M) - \epsilon_3(B).
\end{aligned}$$

The Nash equilibrium region $\boldsymbol{NR}(M,B,B)$ *for* (M,B,B) *strategy is given by*

$$\begin{aligned}\boldsymbol{NR}(M,B,B) = \{(x,y,z) \in \mathbb{R}^3 : x &\leq p + H(M,B,B) \textit{ and} \\ y &\geq p + V(M,B,B) \textit{ and} \\ z &\geq p + D(M,B,B)\},\end{aligned}$$

where the horizontal $H(M,B,B)$, *vertical* $V(M,B,B)$ *and depth* $D(M,B,B)$ *strategic thresholds of the* (M,B,B) *strategy are respectively, given by*

$$H(M,B,B) = \alpha_{11}^M(m_1-1) - \alpha_{12}^B m_2 - \alpha_{13}^B m_3 + \epsilon_1(M) - \epsilon_1(B),$$
$$V(M,B,B) = \alpha_{21}^M m_1 - \alpha_{22}^B(m_2-1) - \alpha_{23}^B m_3 + \epsilon_2(M) - \epsilon_2(B),$$
$$D(M,B,B) = \alpha_{31}^M m_1 - \alpha_{32}^B m_2 - \alpha_{33}^B(m_3-1) + \epsilon_3(M) - \epsilon_3(B).$$

The Nash equilibrium region $\boldsymbol{NR}(M,B,M)$ *for* (M,B,M) *strategy is given by*

$$\begin{aligned}\boldsymbol{NR}(M,B,M) = \{(x,y,z) \in \mathbb{R}^3 : x &\leq p + H(M,B,M) \textit{ and} \\ y &\geq p + V(M,B,M) \textit{ and} \\ z &\leq p + D(M,B,M)\},\end{aligned}$$

where the horizontal $H(M,B,M)$, *vertical* $V(M,B,M)$ *and depth* $D(M,B,M)$ *strategic thresholds of the* (M,B,M) *strategy are respectively, given by*

$$H(M,B,M) = \alpha_{11}^M(m_1-1) - \alpha_{12}^B m_2 + \alpha_{13}^M m_3 + \epsilon_1(M) - \epsilon_1(B),$$
$$V(M,B,M) = \alpha_{21}^M m_1 - \alpha_{22}^B(m_2-1) + \alpha_{23}^M m_3 + \epsilon_2(M) - \epsilon_2(B),$$
$$D(M,B,M) = \alpha_{31}^M m_1 - \alpha_{32}^B m_2 + \alpha_{33}^M(m_3-1) + \epsilon_3(M) - \epsilon_3(B).$$

The Nash equilibrium region $\boldsymbol{NR}(M,M,B)$ *for* (M,M,B) *strategy is given by*

$$\begin{aligned}\boldsymbol{NR}(M,M,B) = \{(x,y,z) \in \mathbb{R}^3 : x &\leq p + H(M,M,B) \textit{ and} \\ y &\leq p + V(M,M,B) \textit{ and} \\ z &\geq p + D(M,M,B)\},\end{aligned}$$

where the horizontal $H(M,M,B)$, *vertical* $V(M,M,B)$ *and depth* $D(M,M,B)$ *strategic thresholds of the* (M,M,B) *strategy are respectively, given by*

$$H(M,M,B) = \alpha_{11}^{M}(m_1 - 1) + \alpha_{12}^{M} m_2 - \alpha_{13}^{B} m_3 + \epsilon_1(M) - \epsilon_1(B),$$
$$V(M,M,B) = \alpha_{21}^{M} m_1 + \alpha_{22}^{M}(m_2 - 1) - \alpha_{23}^{B} m_3 + \epsilon_2(M) - \epsilon_2(B),$$
$$D(M,M,B) = \alpha_{31}^{M} m_1 + \alpha_{32}^{M} m_2 - \alpha_{33}^{B}(m_3 - 1) + \epsilon_3(M) - \epsilon_3(B).$$

The Nash equilibrium region $\boldsymbol{NR}(M,M,M)$ *for* (M,M,M) *strategy is given by*

$$\begin{aligned}\boldsymbol{NR}(M,M,M) = \{(x,y,z) \in \mathbb{R}^3 : x &\leq p + H(M,M,M) \text{ and} \\ y &\leq p + V(M,M,M) \text{ and} \\ z &\leq p + D(M,M,M)\},\end{aligned}$$

where the horizontal $H(M,M,M)$*, vertical* $V(M,M,M)$ *and depth* $D(M,M,M)$ *strategic thresholds of the* (M,M,M) *strategy are respectively, given by*

$$H(M,M,M) = \alpha_{11}^{M}(m_1 - 1) + \alpha_{12}^{M} m_2 + \alpha_{13}^{M} m_3 + \epsilon_1(M) - \epsilon_1(B),$$
$$V(M,M,M) = \alpha_{21}^{M} m_1 + \alpha_{22}^{M}(m_2 - 1) + \alpha_{23}^{M} m_3 + \epsilon_2(M) - \epsilon_2(B),$$
$$D(M,M,M) = \alpha_{31}^{M} m_1 + \alpha_{32}^{M} m_2 + \alpha_{33}^{M}(m_3 - 1) + \epsilon_3(M) - \epsilon_3(B).$$

Chapter 5

Conclusion

We have presented a generalized game theoretical model with finite number of types of homogeneous players. We have characterized all pure strategies (united and separated) and all mixed strategies that form Nash equilibria and determined the corresponding Nash regions for each type of players. A special case of the model is introduced when three types of players are considered. For this case, we have showed geometrically in space all strategies that form Nash equilibria.

We have applied our model in industrial resort sector by considering three types of tourists who will be distributed among two resorts: Beach resort and Mountain resort. We have determine the Nash Equilibrium prices intervals for a given preference for each type of tourists. Such Equilibrium Prices could lead to make monopoly market for one resort while leading the other resort to go bankruptcy. We have also characterized the Nash Equilibrium prices that make duopoly market with high competition in prices.

In the future we will publish this work in a good and related journal. In addition, this work could be extended in the future by considering finite number of decisions instead of allowing only two alternative decisions. Such extension is not trivial to do, however it will generalize the current model. Another extension could be to study the behavioral of human decision by allowing such behaviour for each type of players to follow a certain nonlinear differential equation, so that we end up solving system of nonlinear differential equations using numerical tools.

Appendices

A Appendix A: Special cases for Lemma 3.9

Here are two cases of Lemma 3.9, when we have two types and three types of players.

$$
\begin{aligned}
If\ \ n = 2 \implies\ & S_U(i_1) \in \{d_1, d_2\} \implies l_1^d \in \{0, m_1\}\ \ \forall d \in D \\
& S_U(i_2) \in \{d_1, d_2\} \implies l_2^d \in \{0, m_2\}\ \ \forall d \in D \\
\implies\ & \overline{\boldsymbol{S}_U^2} = \{(0,0), (0,m_2), (m_1,0), (m_1,m_2)\} \\
& \|\overline{\boldsymbol{S}_U^2}\| = 4 = 2^2.
\end{aligned}
$$

$$
\begin{aligned}
If\ \ n = 3 \implies\ & S_U(i_1) \in \{d_1, d_2\} \implies l_1^d \in \{0, m_1\}\ \ \forall d \in D \\
& S_U(i_2) \in \{d_1, d_2\} \implies l_2^d \in \{0, m_2\}\ \ \forall d \in D \\
& S_U(i_3) \in \{d_1, d_2\} \implies l_3^d \in \{0, m_3\}\ \ \forall d \in D \\
\implies\ & \overline{\boldsymbol{S}_U^n} = \{(0,0,0), (0,0,m_3), (0,m_2,0), (0,m_2,m_3), (m_1,0,m_3), \\
& \quad (m_1,m_2,m_3), (m_1,m_2,0), (m_1,0,0)\} \\
& \|\overline{\boldsymbol{S}_U^n}\| = 8 = 2^3.
\end{aligned}
$$

B Appendix B: Two cases for Theorem 3.12

We show the following two cases of Theorem 3.12, when we have two types and three types of players.

$$If\ \ n = 1 \quad \Longrightarrow \quad S_U = (l_1)\ \ and\ \ l_1 \in \{0, m_1\}$$

$if \quad l_1 = m_1$ then the players of type t_1 make decision d_1 iff

$\Pi_1(d_1, m_1) \geq \Pi_1(d_2, m_1 - 1)$;

substituting the values of the utility from (3.5) and (3.6)

and rearrange terms, we get

$x_1 \geq -\alpha_{11}^{d_1}(m_1 - 1) + \epsilon_1(d_2) - \epsilon_1(d_1)$

using (3.11), one can show that

$-\alpha_{11}^{d_1}(m_1 - 1) + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(m_1)$

Hence, $x_1 \geq X_1(m_1)$.

and if $\quad l_1 = 0$ then the players of type t_1 make decision d_2 iff

$\Pi_1(d_2, 0) \geq \Pi_1(d_1, 1)$;

substituting the values of the utility from (3.5) and (3.6)

and rearrange terms, we get

$x_1 \leq \alpha_{11}^{d_2}(m_1 - 1)$.

using (3.12), one can show that

$\alpha_{11}^{d_2}(m_1 - 1) + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(0)$.

Hence, $x_1 \leq X_1(0)$.

$$If\ \ n = 2 \Longrightarrow S_U = (l_1, l_2)\ \ and\ \ l_j \in \{0, m_j\},\ \ j \in \{1, 2\}$$

$$if\ \ l_1 = m_1\ \ and\ \ if\ \ l_2 = m_2$$

$$\begin{cases} \text{then players of type } t_1 \text{ make decision } d_1 \text{ iff} \\ \Pi_1(d_1, m_1, m_2) \geq \Pi_1(d_2, m_1 - 1, m_2); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get } x_1 \geq -\alpha_{11}^{d_1}(m_1 - 1) - \alpha_{12}^{d_1} m_2 + \epsilon_1(d_2) - \epsilon_1(d_1) \\ \text{using (3.11), one can show that} \\ -\alpha_{11}^{d_1}(m_1 - 1) - \alpha_{12}^{d_1} m_2 + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(m_1, m_2) \\ Hence,\ \ x_1 \geq X_1(m_1, m_2) \\ \text{and players of type } t_2 \text{ make decision } d_1 \text{ iff} \\ \Pi_2(d_1, m_1, m_2) \geq \Pi_2(d_2, m_1, m_2 - 1); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get } x_2 \geq -\alpha_{22}^{d_1}(m_2 - 1) - \alpha_{21}^{d_1} m_1 + \epsilon_2(d_2) - \epsilon_2(d_1), \\ \text{using (3.11), one can show that} \\ -\alpha_{22}^{d_1}(m_2 - 1) - \alpha_{21}^{d_1} m_1 + \epsilon_2(d_2) - \epsilon_2(d_1) = X_2(m_1, m_2) \\ Hence,\ \ x_2 \geq X_2(m_1, m_2), \end{cases}$$

$$or\ \ if\ \ l_2 = 0$$

$$\begin{cases} \text{then players of type } t_1 \text{ make decision } d_1 \text{ iff} \\ \Pi_1(d_1, m_1, 0) \geq \Pi_1(d_2, m_1 - 1, 0); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get } x_1 \geq -\alpha_{11}^{d_1}(m_1 - 1) + \alpha_{12}^{d_2} m_2 + \epsilon_1(d_2) - \epsilon_1(d_1), \\ \text{using (3.11), one can show that} \\ -\alpha_{11}^{d_1}(m_1 - 1) + \alpha_{12}^{d_2} m_2 + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(m_1, 0) \\ Hence,\ \ x_1 \geq X_1(m_1, 0) \\ \text{and players of type } t_2 \text{ make decision } d_2 \text{ iff} \\ \Pi_2(d_2, m_1, 0) \geq \Pi_2(d_1, m_1, 1); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get } x_2 \leq \alpha_{22}^{d_2}(m_2 - 1) - \alpha_{21}^{d_1} m_1 + \epsilon_2(d_2) - \epsilon_2(d_1) = X_2(m_1, 0) \\ \text{using (3.12), one can show that} \\ \alpha_{22}^{d_2}(m_2 - 1) - \alpha_{21}^{d_1} m_1 + \epsilon_2(d_2) - \epsilon_2(d_1) = X_2(m_1, 0) \\ Hence,\ \ x_2 \leq X_2(m_1, 0) \end{cases}$$

$if \;\; l_1 = 0 \;\; and \;\; if \;\; l_2 = m_2$

$$\begin{cases} \text{then players of type } t_1 \text{ make decision } d_2 \text{ iff} \\ \Pi_1(d_2, 0, m_2) \geq \Pi_1(d_1, 1, m_2); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get } x_1 \leq \alpha_{11}^{d_2}(m_1 - 1) - \alpha_{12}^{d_1} m_2 + \epsilon_1(d_2) - \epsilon_1(d_1) \\ \text{using (3.12), one can show that} \\ \alpha_{11}^{d_2}(m_1 - 1) - \alpha_{12}^{d_1} m_2 + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(0, m_2) \\ \text{and players of type } t_2 \text{ make decision } d_1 \text{ iff} \\ \Pi_2(d_1, 0, m_2) \geq \Pi_2(d_2, 0, m_2 - 1); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get } x_2 \geq -\alpha_{22}^{d_2}(m_2 - 1) + \alpha_{21}^{d_2} m_1 + \epsilon_2(d_2) - \epsilon_2(d_1) \\ \text{using (3.11), one can show that} \\ -\alpha_{22}^{d_2}(m_2 - 1) + \alpha_{21}^{d_2} m_1 + \epsilon_2(d_2) - \epsilon_2(d_1) = X_2(0, m_2) \\ Hence, \;\; x_2 \geq X_2(0, m_2) \end{cases}$$

$or \;\; if \;\; l_2 = 0$

$$\begin{cases} \text{then players of type } t_1 \text{ make decision } d_2 \text{ iff} \\ \Pi_1(d_2, 0, 0) \geq \Pi_1(d_1, 1, 0); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get } x_1 \leq \alpha_{11}^{d_2}(m_1 - 1) + \alpha_{12}^{d_2} m_2 + \epsilon_1(d_2) - \epsilon_1(d_1) \\ \text{using (3.12), one can show that} \\ \alpha_{11}^{d_2}(m_1 - 1) + \alpha_{12}^{d_2} m_2 + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(0, 0) \\ Hence, x_1 \leq X_1(0, 0) \\ \text{and players of type } t_2 \text{ make decision } d_2 \text{ iff} \\ \Pi_2(d_2, 0, 0) \geq \Pi_2(d_1, 0, 1); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get } x_2 \leq \alpha_{22}^{d_2}(m_2 - 1) - \alpha_{21}^{d_1} m_1 + \epsilon_2(d_2) - \epsilon_2(d_1), \\ \text{using (3.12), one can show that} \\ \alpha_{22}^{d_2}(m_2 - 1) - \alpha_{21}^{d_1} m_1 + \epsilon_2(d_2) - \epsilon_2(d_1) = X_2(0, 0). \\ Hence, \;\; x_2 \leq X_2(0, 0). \end{cases}$$

If $n = 3 \Longrightarrow S_U = (l_1, l_2, l_3)$ and $l_j \in \{0, m_j\},\ \ j \in \{1, 2, 3\}$

if $l_1 = m_1$ and $l_2 = m_2$ and if $l_3 = m_3$

then players of type t_1 make decision d_1 iff

$\Pi_1(d_1, m_1, m_2, m_3) \geq \Pi_1(d_2, m_1 - 1, m_2, m_3)$;

substituting the values of the utility from (3.5) and (3.6) and rearrange terms, we get

$x_1 \geq -\alpha_{11}^{d_1}(m_1 - 1) - \alpha_{12}^{d_1} m_2 - \alpha_{13}^{d_1} + \epsilon_1(d_2) - \epsilon_1(d_1)$

using (3.11), one can show that

$-\alpha_{11}^{d_1}(m_1 - 1) - \alpha_{12}^{d_1} m_2 - \alpha_{13}^{d_1} + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(m_1, m_2, m_3)$.

Hence, $x_1 \geq X_1(m_1, m_2, m_3)$.

and players of type t_2 make decision d_1 iff

$\Pi_2(d_1, m_1, m_2, m_3) \geq \Pi_2(d_2, m_1, m_2 - 1, m_3)$;

substituting the values of the utility from (3.5) and (3.6) and rearrange terms, we get

$x_2 \geq -\alpha_{22}^{d_1}(m_2 - 1) - \alpha_{21}^{d_1} m_1 - \alpha_{23}^{d_1} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1)$

using (3.11), one can show that

$-\alpha_{22}^{d_1}(m_2 - 1) - \alpha_{21}^{d_1} m_1 - \alpha_{23}^{d_1} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1) = X_2(m_1, m_2, m_3)$

$Hence,\ \ x_2 \geq X_2(m_1, m_2, m_3)$

and players of type t_3 make decision d_1 iff

$\Pi_3(d_1, m_1, m_2, m_3) \geq \Pi_3(d_2, m_1, m_2, m_3 - 1)$;

substituting the values of the utility from (3.5) and (3.6) and rearrange terms, we get

$x_3 \geq -\alpha_{33}^{d_1}(m_3 - 1) - \alpha_{31}^{d_1} m_1 - \alpha_{32}^{d_1} m_2 + \epsilon_3(d_2) - \epsilon_3(d_1)$,

using (3.11), one can show that

$-\alpha_{33}^{d_1}(m_3 - 1) - \alpha_{31}^{d_1} m_1 - \alpha_{32}^{d_1} m_2 + \epsilon_3(d_2) - \epsilon_3(d_1) = X_3(m_1, m_2, m_3)$.

$Hence,\ \ x_3 \geq X_3(m_1, m_2, m_3)$

if $l_1 = m_1$ and $l_2 = m_2$ and if $l_3 = 0$

$$\begin{cases}
\text{then players of type } t_1 \text{ make decision } d_1 \text{ iff} \\
\Pi_1(d_1, m_1, m_2, 0) \geq \Pi_1(d_2, m_1 - 1, m_2, 0); \\
\text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\
\text{we get} \\
x_1 \geq -\alpha_{11}^{d_1}(m_1 - 1) - \alpha_{12}^{d_1} m_2 + \alpha_{13}^{d_2} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(m_1, m_2, 0) \\
\text{using (3.11), one can show that} \\
-\alpha_{11}^{d_1}(m_1 - 1) - \alpha_{12}^{d_1} m_2 + \alpha_{13}^{d_2} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(m_1, m_2, 0). \\
Hence, \;\; x_1 \geq X_1(m_1, m_2, 0) \\
\text{and players of type } t_2 \text{ make decision } d_1 \text{ iff} \\
\Pi_2(d_1, m_1, m_2, 0) \geq \Pi_2(d_2, m_1, m_2 - 1, 0); \\
\text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\
\text{we get} \\
x_2 \geq -\alpha_{22}^{d_1}(m_2 - 1) - \alpha_{21}^{d_1} m_1 + \alpha_{23}^{d_2} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1), \\
\text{using (3.11), one can show that} \\
-\alpha_{22}^{d_1}(m_2 - 1) - \alpha_{21}^{d_1} m_1 + \alpha_{23}^{d_2} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1) = X_2(m_1, m_2, 0) \\
Hence, \;\; x_2 \geq X_2(m_1, m_2, 0) \\
\text{and players of type } t_3 \text{ make decision } d_2 \text{ iff} \\
\Pi_3(d_2, m_1, m_2, 0) \geq \Pi_3(d_1, m_1, m_2, 1); \\
\text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\
\text{we get} \\
x_3 \leq \alpha_{33}^{d_2}(m_3 - 1) - \alpha_{31}^{d_1} m_1 - \alpha_{32}^{d_1} m_2 + \epsilon_3(d_2) - \epsilon_3(d_1), \\
\text{using (3.12), one can show that} \\
\alpha_{33}^{d_2}(m_3 - 1) - \alpha_{31}^{d_1} m_1 - \alpha_{32}^{d_1} m_2 + \epsilon_3(d_2) - \epsilon_3(d_1) = X_3(m_1, m_2, 0). \\
Hence, x_3 \leq X_3(m_1, m_2, 0)
\end{cases}$$

if $l_1 = m_1$ and $l_2 = 0$ $then\ \ if\ \ l_3 = m_3$

then players of type t_1 make decision d_1 iff

$\Pi_1(d_1, m_1, 0, m_3) \geq \Pi_1(d_2, m_1 - 1, 0, m_3)$;

substituting the values of the utility from (3.5) and (3.6) and rearrange terms, we get

$x_1 \geq -\alpha_{11}^{d_1}(m_1 - 1) + \alpha_{12}^{d_2}m_2 - \alpha_{13}^{d_1}m_3 + \epsilon_1(d_2) - \epsilon_1(d_1)$,

using (3.11), one can show that

$-\alpha_{11}^{d_1}(m_1 - 1) + \alpha_{12}^{d_2}m_2 - \alpha_{13}^{d_1}m_3 + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(m_1, 0, m_3)$.

$Hence, x_1 \geq X_1(m_1, 0, m_3)$

and players of type t_2 make decision d_2 iff

$\Pi_2(d_2, m_1, 0, m_3) \geq \Pi_2(d_1, m_1, 1, m_3)$;

substituting the values of the utility from (3.5) and (3.6) and rearrange terms, we get

$x_2 \leq \alpha_{22}^{d_2}(m_2 - 1) - \alpha_{21}^{d_1}m_1 - \alpha_{23}^{d_1}m_3 + \epsilon_2(d_2) - \epsilon_2(d_1)$,

using (3.12), one can show that

$\alpha_{22}^{d_2}(m_2 - 1) - \alpha_{21}^{d_1}m_1 - \alpha_{23}^{d_1}m_3 + \epsilon_2(d_2) - \epsilon_2(d_1) = X_2(m_1, 0, m_3)$.

$Hence, x_2 \leq X_2(m_1, 0, m_3)$

and players of type t_3 make decision d_1 iff

$\Pi_3(d_1, m_1, 0, m_3) \geq \Pi_3(d_2, m_1, 0, m_3 - 1)$;

substituting the values of the utility from (3.5) and (3.6) and rearrange terms, we get

$x_3 \geq -\alpha_{33}^{d_1}(m_3 - 1) - \alpha_{31}^{d_1}m_1 + \alpha_{32}^{d_2}m_2 + \epsilon_3(d_2) - \epsilon_3(d_1)$,

using (3.11), one can show that

$-\alpha_{33}^{d_1}(m_3 - 1) - \alpha_{31}^{d_1}m_1 + \alpha_{32}^{d_2}m_2 + \epsilon_3(d_2) - \epsilon_3(d_1) = X_3(m_1, 0, m_3)$.

$Hence,\ \ x_3 \geq X_3(m_1, 0, m_3)$.

if $l_1 = m_1$ and $l_2 = 0$ $then\ if\ l_3 = 0$

$$\begin{cases} \text{then players of type } t_1 \text{ make decision } d_1 \text{ iff} \\ \Pi_1(d_1, m_1, 0, 0) \geq \Pi_1(d_2, m_1 - 1, 0, 0); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get} \\ x_1 \geq -\alpha_{11}^{d_1}(m_1 - 1) + \alpha_{12}^{d_2} m_2 + \alpha_{13}^{d_2} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1), \\ \text{using (3.11), one can show that} \\ -\alpha_{11}^{d_1}(m_1 - 1) + \alpha_{12}^{d_2} m_2 + \alpha_{13}^{d_2} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(m_1, 0, 0). \\ Hence, x_1 \geq X_1(m_1, 0, 0) \\ \text{and players of type } t_2 \text{ make decision } d_2 \text{ iff} \\ \Pi_2(d_2, m_1, 0, 0) \geq \Pi_2(d_1, m_1, 1, 0); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get} \\ x_2 \leq \alpha_{22}^{d_2}(m_2 - 1) - \alpha_{21}^{d_1} m_1 + \alpha_{23}^{d_2} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1), \\ \text{using (3.12), one can show that} \\ \alpha_{22}^{d_2}(m_2 - 1) - \alpha_{21}^{d_1} m_1 + \alpha_{23}^{d_2} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1) = X_2(m_1, 0, 0). \\ Hence, x_2 \leq X_2(m_1, 0, 0). \\ \text{and players of type } t_3 \text{ make decision } d_2 \text{ iff} \\ \Pi_3(d_2, m_1, 0, 0) \geq \Pi_3(d_2, m_1, 0, 1); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get} \\ x_3 \leq \alpha_{33}^{d_2}(m_3 - 1) - \alpha_{31}^{d_1} m_1 + \alpha_{32}^{d_2} m_2 + \epsilon_3(d_2) - \epsilon_3(d_1), \\ \text{using (3.12), one can show that} \\ \alpha_{33}^{d_2}(m_3 - 1) - \alpha_{31}^{d_1} m_1 + \alpha_{32}^{d_2} m_2 + \epsilon_3(d_2) - \epsilon_3(d_1) = X_3(m_1, 0, 0). \\ Hence,\ \ x_3 \leq X_3(m_1, 0, 0) \end{cases}$$

if $l_1 = 0$ and $l_2 = m_2$ and if $l_3 = m_3$

$$\begin{cases} \text{then players of type } t_1 \text{ make decision } d_2 \text{ iff} \\ \Pi_1(d_2, 0, m_2, m_3) \geq \Pi_1(d_1, 1, m_2, m_3); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get} \\ x_1 \leq \alpha_{11}^{d_2}(m_1 - 1) - \alpha_{12}^{d_1} m_2 - \alpha_{13}^{d_1} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1), \\ \text{using (3.12), one can show that} \\ \alpha_{11}^{d_2}(m_1 - 1) - \alpha_{12}^{d_1} m_2 - \alpha_{13}^{d_1} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(0, m_2, m_3). \\ Hence, x_1 \leq X_1(0, m_2, m_3). \\ \text{and players of type } t_2 \text{ make decision } d_1 \text{ iff} \\ \Pi_2(d_1, 0, m_2, m_3) \geq \Pi_2(d_2, 0, m_2 - 1, m_3); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get} \\ x_2 \geq -\alpha_{22}^{d_1}(m_2 - 1) + \alpha_{21}^{d_2} m_1 - \alpha_{23}^{d_1} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1), \\ \text{using (3.11), one can show that} \\ -\alpha_{22}^{d_1}(m_2 - 1) + \alpha_{21}^{d_2} m_1 - \alpha_{23}^{d_1} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1) = X_2(0, m_2, m_3). \\ Hence, \;\; x_2 \geq X_2(0, m_2, m_3). \\ \text{and players of type } t_3 \text{ make decision } d_1 \text{ iff} \\ \Pi_3(d_1, 0, m_2, m_3) \geq \Pi_3(d_2, 0, m_2, m_3 - 1); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get} \\ x_3 \geq -\alpha_{33}^{d_1}(m_3 - 1) + \alpha_{31}^{d_2} m_1 - \alpha_{32}^{d_1} m_2 + \epsilon_3(d_2) - \epsilon_3(d_1), \\ \text{using (3.11), one can show that} \\ -\alpha_{33}^{d_1}(m_3 - 1) + \alpha_{31}^{d_2} m_1 - \alpha_{32}^{d_1} m_2 + \epsilon_3(d_2) - \epsilon_3(d_1) = X_3(0, m_2, m_3). \\ Hence, \;\; x_3 \geq X_3(0, m_2, m_3) \end{cases}$$

if $l_1 = 0$ and $l_2 = m_2$ and if $l_3 = 0$

$$\begin{cases} \text{then players of type } t_1 \text{ make decision } d_2 \text{ iff} \\ \Pi_1(d_2, 0, m_2, 0) \geq \Pi_1(d_1, 1, m_2, 0); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get} \\ x_1 \leq \alpha_{11}^{d_2}(m_1 - 1) - \alpha_{12}^{d_1}m_2 + \alpha_{13}^{d_2}m_3 + \epsilon_1(d_2) - \epsilon_1(d_1), \\ \text{using (3.12), one can show that} \\ \alpha_{11}^{d_2}(m_1 - 1) - \alpha_{12}^{d_1}m_2 + \alpha_{13}^{d_2}m_3 + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(0, m_2, 0). \\ Hence, x_1 \leq X_1(0, m_2, 0). \\ \text{and players of type } t_2 \text{ make decision } d_1 \text{ iff} \\ \Pi_2(d_1, 0, m_2, 0) \geq \Pi_2(d_2, 0, m_2 - 1, 0); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get} \\ x_2 \geq -\alpha_{22}^{d_1}(m_2 - 1) + \alpha_{21}^{d_2}m_1 + \alpha_{23}^{d_2}m_3 + \epsilon_2(d_2) - \epsilon_2(d_1), \\ \text{using (3.11), one can show that} \\ -\alpha_{22}^{d_1}(m_2 - 1) + \alpha_{21}^{d_2}m_1 + \alpha_{23}^{d_2}m_3 + \epsilon_2(d_2) - \epsilon_2(d_1) = X_2(0, m_2, 0). \\ Hence, \;\; x_2 \geq X_2(0, m_2, 0). \\ \text{and players of type } t_3 \text{ make decision } d_2 \text{ iff} \\ \Pi_3(d_2, 0, m_2, 0) \geq \Pi_3(d_1, 0, m_2, 1); \\ \text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\ \text{we get} \\ x_3 \leq \alpha_{33}^{d_2}(m_3 - 1) + \alpha_{31}^{d_2}m_1 - \alpha_{32}^{d_1}m_2 + \epsilon_3(d_2) - \epsilon_3(d_1), \\ \text{using (3.12), one can show that} \\ \alpha_{33}^{d_2}(m_3 - 1) + \alpha_{31}^{d_2}m_1 - \alpha_{32}^{d_1}m_2 + \epsilon_3(d_2) - \epsilon_3(d_1) = X_3(0, m_2, 0). \\ Hence, x_3 \leq X_3(0, m_2, 0) \end{cases}$$

if $l_1 = 0$ and $l_2 = 0$ $then\ \ if\ \ l_3 = m_3$

$$\begin{cases}
\text{then players of type } t_1 \text{ make decision } d_2 \text{ iff} \\
\Pi_1(d_2, 0, 0, m_3) \geq \Pi_1(d_1, 1, 0, m_3); \\
\text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\
\text{we get} \\
x_1 \leq \alpha_{11}^{d_2}(m_1 - 1) + \alpha_{12}^{d_2} m_2 - \alpha_{13}^{d_1} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1), \\
\text{using (3.12), one can show that} \\
\alpha_{11}^{d_2}(m_1 - 1) + \alpha_{12}^{d_2} m_2 - \alpha_{13}^{d_1} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(0, 0, m_3). \\
Hence,\ \ x_1 \leq X_1(0, 0, m_3) \\
\text{and players of type } t_2 \text{ make decision } d_2 \text{ iff} \\
\Pi_2(d_2, 0, 0, m_3) \geq \Pi_2(d_1, 0, 1, m_3); \\
\text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\
\text{we get} \\
x_2 \leq \alpha_{22}^{d_2}(m_2 - 1) + \alpha_{21}^{d_2} m_1 - \alpha_{23}^{d_1} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1), \\
\text{using (3.12), one can show that} \\
\alpha_{22}^{d_2}(m_2 - 1) + \alpha_{21}^{d_2} m_1 - \alpha_{23}^{d_1} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1) = X_2(0, 0, m_3). \\
Hence,\ \ x_2 \leq X_2(0, 0, m_3). \\
\text{and players of type } t_3 \text{ make decision } d_1 \text{ iff} \\
\Pi_3(d_1, 0, 0, m_3) \geq \Pi_3(d_2, 0, 0, m_3 - 1); \\
\text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\
\text{we get} \\
x_3 \geq -\alpha_{33}^{d_1}(m_3 - 1) + \alpha_{31}^{d_2} m_1 + \alpha_{32}^{d_2} m_2 + \epsilon_3(d_2) - \epsilon_3(d_1), \\
\text{using (3.11), one can show that} \\
-\alpha_{33}^{d_1}(m_3 - 1) + \alpha_{31}^{d_2} m_1 + \alpha_{32}^{d_2} m_2 + \epsilon_3(d_2) - \epsilon_3(d_1) = X_3(0, 0, m_3). \\
Hence,\ \ x_3 \geq X_3(0, 0, m_3).
\end{cases}$$

if $l_1 = 0$ and $l_2 = m_2$ and if $l_3 = 0$

$$\begin{cases}
\text{then players of type } t_1 \text{ make decision } d_2 \text{ iff} \\
\Pi_1(d_2, 0, 0, 0) \geq \Pi_1(d_1, 1, 0, 0); \\
\text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\
\text{we get} \\
x_1 \leq \alpha_{11}^{d_2}(m_1 - 1) + \alpha_{12}^{d_2} m_2 + \alpha_{13}^{d_2} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1), \\
\text{using (3.12), one can show that} \\
\alpha_{11}^{d_2}(m_1 - 1) + \alpha_{12}^{d_2} m_2 + \alpha_{13}^{d_2} m_3 + \epsilon_1(d_2) - \epsilon_1(d_1) = X_1(0, 0, 0). \\
Hence, \;\; x_1 \leq X_1(0, 0, 0). \\
\text{and players of type } t_2 \text{ make decision } d_2 \text{ iff} \\
\Pi_2(d_2, 0, 0, 0) \geq \Pi_2(d_1, 0, 1, 0); \\
\text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\
\text{we get} \\
x_2 \leq \alpha_{22}^{d_2}(m_2 - 1) + \alpha_{21}^{d_2} m_1 + \alpha_{23}^{d_2} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1), \\
\text{using (3.12), one can show that} \\
\alpha_{22}^{d_2}(m_2 - 1) + \alpha_{21}^{d_2} m_1 + \alpha_{23}^{d_2} m_3 + \epsilon_2(d_2) - \epsilon_2(d_1) = X_2(0, 0, 0). \\
Hence, \;\; x_2 \leq X_2(0, 0, 0). \\
\text{and players of type } t_3 \text{ make decision } d_2 \text{ iff} \\
\Pi_3(d_2, 0, 0, 0) \geq \Pi_3(d_2, 0, 0, 1); \\
\text{substituting the values of the utility from (3.5) and (3.6) and rearrange terms,} \\
\text{we get} \\
x_3 \leq \alpha_{33}^{d_2}(m_3 - 1) + \alpha_{31}^{d_2} m_1 + \alpha_{32}^{d_2} m_2 + \epsilon_3(d_2) - \epsilon_3(d_1), \\
\text{using (3.12), one can show that} \\
\alpha_{33}^{d_2}(m_3 - 1) + \alpha_{31}^{d_2} m_1 + \alpha_{32}^{d_2} m_2 + \epsilon_3(d_2) - \epsilon_3(d_1) = X_3(0, 0, 0). \\
Hence, \;\; x_3 \leq X_3(0, 0, 0).
\end{cases}$$

C Appendix C: Special cases of Theorem 3.12

We have here more special cases of Theorem 3.12.

Lemma .1. *A united strategy $S_U = (0, m_2, \cdots, m_n) \in \mathbb{R}^n$ is Nash Equilibrium iff $x \in \boldsymbol{NR}(0, m_2, \cdots, m_n)$, where the Nash region $\boldsymbol{NR}(0, m_2, \cdots, m_n)$ is given by*

$$\begin{aligned} \boldsymbol{NR}(0, m_2, \cdots, m_n) = \{x \in \mathbb{R}^n : x_1 &\leq X_1(0, m_2, \cdots, m_n) \text{ and} \\ x_2 &\geq X_2(0, m_2, \cdots, m_n) \text{ and} \\ &\vdots \\ x_n &\geq X_n(0, m_2, \cdots, m_n)\}. \end{aligned}$$

Proof. Consider the following united strategy: $S_U = (0, m_2, \cdots, m_n) \in \mathbb{R}^n$, where all players with type t_1 make the decision d_2, and all players of types $t_2, t_3, \cdots, t_n$ make decision d_1. The united strategy $S_U = (0, m_2, \cdots, m_n)$ is NE iff

$$\begin{aligned} \Pi_1(d_1; 1, m_2, \cdots, m_n) &\leq \Pi_1(d_2; 0, m_2, \cdots, m_n) \text{ and} \\ \Pi_2(d_1; 0, m_2, \cdots, m_n) &\geq \Pi_2(d_2; 0, m_2 - 1, \cdots, m_n) \text{ and} \\ &\vdots \\ \Pi_n(d_1; 0, m_2, \cdots, m_n) &\geq \Pi_n(d_2; 0, m_2, \cdots, m_n - 1). \end{aligned} \tag{1}$$

Substituting the utility functions given in (3.5) and (3.6) in inequality (1), then we obtain

$$\begin{aligned} &\omega_1^{d_1} + \alpha_{11}^{d_1}(1-1) + \alpha_{12}^{d_1}m_2 + \cdots + \alpha_{1n}^{d_1}m_n + \epsilon_1(d_1) \leq \\ &\omega_1^{d_2} + \alpha_{11}^{d_2}(m_1 - 1) + \alpha_{12}^{d_2}(m_2 - m_2) + \cdots + \alpha_{1n}^{d_2}(m_n - m_n) + \epsilon_1(d_2) \text{ and} \\ &\omega_2^{d_1} + \alpha_{21}^{d_1}(0) + \alpha_{22}^{d_1}(m_2 - 1) + \cdots + \alpha_{2n}^{d_1}m_n + \epsilon_2(d_1) \geq \\ &\omega_2^{d_2} + \alpha_{21}^{d_2}(m_1 - 0) + \alpha_{22}^{d_2}(m_2 - (m_2 - 1) - 1) + \cdots + \alpha_{2n}^{d_2}(m_n - m_n) + \epsilon_2(d_2) \text{ and} \\ &\vdots \\ &\omega_n^{d_1} + \alpha_{n1}^{d_1}(0) + \alpha_{n2}^{d_1}m_2 + \cdots + \alpha_{nn}^{d_1}(m_n - 1) + \epsilon_n(d_1) \geq \\ &\omega_n^{d_2} + \alpha_{n1}^{d_2}(m_1 - 0) + \alpha_{n2}^{d_2}(m_2 - m_2) + \cdots + \alpha_{nn}^{d_2}(m_n - (m_n - 1) - 1) + \epsilon_n(d_2). \end{aligned} \tag{2}$$

Rearrange the previous inequalities (2) we get

$$\begin{aligned}
\omega_1^{d_1} - \omega_1^{d_2} &\leq \alpha_{11}^{d_2}(m_1 - 1) - \alpha_{12}^{d_1} m_2 - \cdots - \alpha_{1n}^{d_1} m_n + \epsilon_1(d_2) - \epsilon_1(d_1) \ \textit{and} \\
\omega_2^{d_1} - \omega_2^{d_2} &\geq \alpha_{21}^{d_2} m_1 - \alpha_{22}^{d_2}(m_2 - 1) - \cdots - \alpha_{2n}^{d_1} m_n + \epsilon_2(d_2) - \epsilon_2(d_1) \ \textit{and} \\
&\vdots \\
\omega_n^{d_1} - \omega_n^{d_2} &\geq \alpha_{n1}^{d_2} m_1 - \alpha_{n2}^{d_1} m_2 - \cdots - \alpha_{nn}^{d_1}(m_n - 1) + \epsilon_n(d_2) - \epsilon_n(d_1).
\end{aligned}$$

Substituting the values of the relative decisions from Definition 3.5, the last inequalities simplifying to

$$\begin{aligned}
x_1 &\leq X_1(0, m_2, \cdots, m_n) \ \textit{and} \\
x_2 &\geq X_2(0, m_2, \cdots, m_n) \ \textit{and} \\
&\vdots \\
x_n &\geq X_n(0, m_2, \cdots, m_n),
\end{aligned}$$

where the strategic thresholds of the $S_U = (0, m_2, \cdots, m_n)$ strategy are, respectively, given by

$$\begin{aligned}
X_1(0, m_2, \cdots, m_n) &= \alpha_{11}^{d_2}(m_1 - 1) - \alpha_{12}^{d_1} m_2 - \cdots - \alpha_{1n}^{d_1} m_n + \epsilon_1(d_2) - \epsilon_1(d_1) \\
X_2(0, m_2, \cdots, m_n) &= \alpha_{21}^{d_2} m_1 - \alpha_{22}^{d_2}(m_2 - 1) - \cdots - \alpha_{2n}^{d_1} m_n + \epsilon_2(d_2) - \epsilon_2(d_1) \\
&\vdots \\
X_n(0, m_2, \cdots, m_n) &= \alpha_{n1}^{d_2} m_1 - \alpha_{n2}^{d_1} m_2 - \cdots - \alpha_{nn}^{d_1}(m_n - 1) + \epsilon_n(d_2) - \epsilon_n(d_1).
\end{aligned}$$

Hence, the corresponding Nash region is

$$\begin{aligned}
\boldsymbol{NR}(0, m_2, \cdots, m_n) = \{x \in \mathbb{R}^n : x_1 &\leq X_1(0, m_2, \cdots, m_n) \ \textit{and} \\
x_2 &\geq X_2(0, m_2, \cdots, m_n) \ \textit{and} \\
&\vdots \\
x_n &\geq X_n(0, m_2, \cdots, m_n)\}.
\end{aligned}$$

□

Lemma .2. *A united strategy* $S_U = (m_1, m_2, 0\cdots, 0) \in \mathbb{R}^n$ *is Nash Equilibrium iff* $x \in \boldsymbol{NR}(m_1, m_2, 0\cdots, 0)$*, where the Nash region* $\boldsymbol{NR}(m_1, m_2, 0\cdots, 0)$ *is given by*

$$\begin{aligned} \boldsymbol{NR}(m_1, m_2, 0\cdots, 0) = \{x \in \mathbb{R}^n : x_1 &\geq X_1(m_1, m_2, 0\cdots, 0) \ \textit{and} \\ x_2 &\geq X_2(m_1, m_2, 0\cdots, 0) \ \textit{and} \\ x_3 &\leq X_3(m_1, m_2, 0\cdots, 0) \ \textit{and} \\ &\vdots \\ x_n &\leq X_n(m_1, m_2, 0\cdots, 0)\}. \end{aligned}$$

Proof. Consider the following united strategy: $S_U = (m_1, m_2, 0\cdots, 0) \in \mathbb{R}^n$, where all players of types t_1 and t_2 make the decision d_1 and all players of types $t_3, t_4, \cdots, t_n$ make the decision d_2. The united strategy $S_U = (m_1, m_2, 0\cdots, 0)$ is NE iff

$$\begin{aligned} \Pi_1(d_1; m_1, m_2, 0, 0, \cdots, 0) &\geq \Pi_1(d_2; m_1 - 1, m_2, 0, \cdots, 0) \ \textit{and} \\ \Pi_2(d_1; m_1, m_2, 0, 0, \cdots, 0) &\geq \Pi_2(d_2; m_1, m_2 - 1, 0, \cdots, 0) \ \textit{and} \\ \Pi_3(d_1; m_1, m_2, 1, 0, \cdots, 0) &\leq \Pi_3(d_2; m_1, m_2, 0, \cdots, 0) \ \textit{and} \\ &\vdots \\ \Pi_n(d_1; m_1, m_2, 0\cdots, 0, 1) &\leq \Pi_n(d_2; m_1, m_2, 0\cdots, 0). \end{aligned} \tag{3}$$

Substituting the utility functions given in (3.5) and (3.6) in inequality (3), then we obtain

$$
\begin{aligned}
&\omega_1^{d_1} + \alpha_{11}^{d_1}(m_1-1) + \alpha_{12}^{d_1}m_2 + \alpha_{13}^{d_1}(0)\cdots + \alpha_{1n}^{d_1}(0) + \epsilon_1(d_1) \geq \omega_1^{d_2}\\
&+\alpha_{11}^{d_2}(m_1-(m_1-1)-1) + \alpha_{12}^{d_2}(m_2-m_2) + \alpha_{13}^{d_2}(m_3-0) + \cdots + \alpha_{1n}^{d_2}(m_n-0)\\
&+\epsilon_1(d_2)\ \ and\\
&\omega_2^{d_1} + \alpha_{21}^{d_1}m_1 + \alpha_{22}^{d_1}(m_2-1) + \alpha_{23}^{d_1}(0) + \cdots + \alpha_{2n}^{d_1}(0) + \epsilon_2(d_1) \geq \omega_2^{d_2} + \alpha_{21}^{d_2}(m_1-m_1)\\
&+\alpha_{22}^{d_2}(m_2-(m_2-1)-1) + \alpha_{23}^{d_2}(m_3-0) + \cdots + \alpha_{2n}^{d_2}(m_n-0) + \epsilon_2(d_2)\ \ and \qquad (4)\\
&\omega_3^{d_1} + \alpha_{31}^{d_1}m_1 + \alpha_{32}^{d_1}m_2 + \alpha_{33}^{d_1}(1-1)\cdots + \alpha_{3n}^{d_1}(0) + \epsilon_3(d_1) \leq \omega_3^{d_2} + \alpha_{31}^{d_2}(m_1-m_1)\\
&+\alpha_{32}^{d_2}(m_2-m_2) + \alpha_{33}^{d_2}(m_3-1) + \cdots + \alpha_{3n}^{d_2}(m_n-0) + \epsilon_3(d_2)\ \ and\\
&\qquad\vdots\\
&\omega_n^{d_1} + \alpha_{n1}^{d_1}m_1 + \alpha_{n2}^{d_1}m_2 + \alpha_{n3}^{d_1}(0)\cdots + \alpha_{nn}^{d_1}(1-1) + \epsilon_n(d_1) \leq \omega_n^{d_2} + \alpha_{n1}^{d_2}(m_1-m_1)\\
&+\alpha_{n2}^{d_2}(m_2-m_2) + \alpha_{n3}^{d_2}(m_3-0) + \cdots + \alpha_{nn}^{d_2}(m_n-1) + \epsilon_n(d_2).
\end{aligned}
$$

Rearrange the previous inequalities (4), we get

$$
\begin{aligned}
&\omega_1^{d_1} - \omega_1^{d_2} \geq -\alpha_{11}^{d_1}(m_1-1) - \alpha_{12}^{d_1}m_2 + \alpha_{13}^{d_2}m_3 + \cdots + \alpha_{1n}^{d_2}m_n + \epsilon_1(d_2) - \epsilon_1(d_1)\ and\\
&\omega_2^{d_1} - \omega_2^{d_2} \geq -\alpha_{21}^{d_1}m_1 - \alpha_{22}^{d_1}(m_2-1) + \alpha_{23}^{d_2}m_3 + \cdots + \alpha_{2n}^{d_2}m_n + \epsilon_2(d_2) - \epsilon_2(d_1)\ and\\
&\omega_3^{d_1} - \omega_3^{d_2} \leq -\alpha_{31}^{d_1}m_1 - \alpha_{32}^{d_1}m_2 + \alpha_{33}^{d_2}(m_3-1) + \cdots + \alpha_{3n}^{d_2}m_n + \epsilon_3(d_2) - \epsilon_3(d_1)\ and\\
&\qquad\vdots\\
&\omega_n^{d_1} - \omega_n^{d_2} \leq -\alpha_{n1}^{d_1}m_1 - \alpha_{n2}^{d_1}m_2 + \alpha_{n3}^{d_2}m_3 + \cdots + \alpha_{nn}^{d_2}(m_n-1) + \epsilon_n(d_2) - \epsilon_n(d_1).
\end{aligned}
$$

Substituting the values of the relative decisions from Definition 3.5, the last inequalities simplifying to

$$
\begin{aligned}
x_1 &\geq X_1(m_1, m_2, 0, \cdots, 0)\ \ and\\
x_2 &\geq X_2(m_1, m_2, 0, \cdots, 0)\ \ and\\
x_3 &\leq X_3(m_1, m_2, 0, \cdots, 0)\ \ and\\
&\ \vdots\\
x_n &\leq X_n(m_1, m_2, 0, \cdots, 0),
\end{aligned}
$$

where the strategic thresholds of the $S_U = (m_1, m_2, 0, \cdots, 0)$ strategy are, respectively, given by

$$\begin{aligned}
X_1(m_1, m_2, 0, \cdots, 0) = & -\alpha_{11}^{d_1}(m_1 - 1) - \alpha_{12}^{d_1}m_2 + \alpha_{13}^{d_2}m_3 + \cdots + \alpha_{1n}^{d_2}m_n \\
& + \epsilon_1(d_2) - \epsilon_1(d_1) \ \ and \\
X_2(m_1, m_2, 0, \cdots, 0) = & -\alpha_{21}^{d_1}m_1 - \alpha_{22}^{d_1}(m_2 - 1) + \alpha_{23}^{d_2}m_3 + \cdots + \alpha_{2n}^{d_2}m_n \\
& + \epsilon_2(d_2) - \epsilon_2(d_1) \ \ and \\
X_3(m_1, m_2, 0, \cdots, 0) = & -\alpha_{31}^{d_1}m_1 - \alpha_{32}^{d_1}m_2 + \alpha_{33}^{d_2}(m_3 - 1) + \cdots + \alpha_{3n}^{d_2}m_n \\
& + \epsilon_3(d_2) - \epsilon_3(d_1) \ \ and \\
& \vdots \\
X_n(m_1, m_2, 0, \cdots, 0) = & -\alpha_{n1}^{d_1}m_1 - \alpha_{n2}^{d_1}m_2 + \alpha_{n3}^{d_2}m_3 + \cdots + \alpha_{nn}^{d_2}(m_n - 1) \\
& + \epsilon_n(d_2) - \epsilon_n(d_1).
\end{aligned}$$

Hence,

$$\begin{aligned}
\boldsymbol{NR}(m_1, m_2, 0 \cdots, 0) = \{x \in \mathbb{R}^n : x_1 & \geq X_1(m_1, m_2, 0, \cdots, 0) \ \ and \\
x_2 & \geq X_2(m_1, m_2, 0, \cdots, 0) \ \ and \\
x_3 & \leq X_3(m_1, m_2, 0, \cdots, 0) \ \ and \\
& \vdots \\
x_n & \leq X_n(m_1, m_2, 0, \cdots, 0)\}.
\end{aligned}$$

□

D Appendix D: Simulation using Mathematica

In this appendix we will show mathematica codes used to draw the figures in three dimensions as they are shown in this Thesis.

Code for Figure 3.2.

```
SetOptions[Plot, BaseStyle -> {FontFamily -> "Times", FontSize -> 14}];
A=Graphics3D[{Arrowheads[Large],
  Arrow[{{-5,-5, -5}, {4, -5,-5}}],Text["x",{5,-5,-5}], Arrow[{{-5,-5, -5},
  {-5, 4, -5}}], Text["y",{-5,5,-5}], Arrow[{{-5,-5, -5}, {-5, -5, 4}}],
  Text["z",{-5,-5,5}]},Boxed->False];
p1 = RegionPlot3D[
   x >-3 && y > -3 && z > -3, {x, -6, 6}, {y, -6, 6}, {z, -6, 6}, Boxed -> False,
   PlotStyle -> Directive[Yellow, Opacity[0.5`]], Mesh -> None,
   AxesEdge -> {{0, 0}, {0, 0}, {0, 0}},
   AxesLabel-> {Style["x",FontSize->16,FontWeight->Bold],
   Style["y",FontSize->16,FontWeight->Bold], Style["z",FontSize->16,
   FontWeight->Bold]}, PlotRange -> {{-6, 5}, {-6, 5}, {-6, 5}},   Ticks -> None];
X=Graphics3D[{ Text[Style["X(d1,d1,d1)",FontSize->12], {-3, -5, -5}, {0, 4}]}];
Y=Graphics3D[{ Text[Style["Y(d1,d1,d1)",FontSize->12], {-5, -3, -5}, {-0.4,-2}]}];
Z=Graphics3D[{ Text[Style["Z(d1,d1,d1)",FontSize->12], {-5, -5, -3}, {1, 2}]}];
pi1=Graphics3D[{PointSize[0.02],Point[{-3,-5,-5}]}];
pi2=Graphics3D[{PointSize[0.02],Point[{-5,-3,-5}]}];
pi3=Graphics3D[{PointSize[0.02],Point[{-5,-5,-3}]}];
Show[A,p1,pi1,pi2,pi3,X,Y,Z]
```

Code for Figure 3.3.

```
SetOptions[Plot,
  BaseStyle -> {FontFamily -> "Times", FontSize -> 14}];
A = Graphics3D[{Arrowheads[Large],
       Arrow[{{0, 0, 0}, {0, 6, 0}}], Text["y", {0, 7, 0}],
      Arrow[{{0, 0, 0}, {0, 0, 6}}], Text["z", {0, 0, 7}]},
   Boxed -> False, AxesEdge -> {{-1, 0}, None, None}];
B = Graphics3D[{Arrowheads[Tiny],
      Arrow[{{0, 0, 0}, {-6, 0, 0}}], Text["x", {-3, -1, 0}]},
   Boxed -> False, AxesEdge -> {{-1, 0}, None, None}];
p2 = RegionPlot3D[
     x < -1 && y > 1 && z > 1, {x, -6, 0}, {y, 0, 6}, {z, 0, 6},
     Boxed -> False, PlotStyle -> Directive[Red, Opacity[0.5`]],
     Mesh -> None, AxesEdge -> {{-1, 0}, None, None}, Axes -> True,
     AxesLabel -> {"x", "y", "z"},  Ticks -> None];
X =Graphics3D[{
    Text[Style["X(d2,d1,d1)", FontSize -> 12], {-1, 0, 0}, {.5,
      1}]}];
Y =Graphics3D[{
    Text[Style["Y(d2,d1,d1)", FontSize -> 12], {0, 1,
      0}, {-1, -1}]}];
Z =Graphics3D[{
    Text[Style["Z(d2,d1,d1)", FontSize -> 12], {0, 0, 1}, {1, -1}]}];
pi1 =Graphics3D[{PointSize[0.02], Point[{-1, 0, 0}]}];
pi2 =Graphics3D[{PointSize[0.02], Point[{0, 1, 0}]}];
pi3 =Graphics3D[{PointSize[0.02], Point[ {0, 0, 1}]}];
Show[A, p2, B, X, Y, Z, pi1, pi2, pi3]
```

Code for Figure 3.4.

```
SetOptions[Plot,
  BaseStyle -> {FontFamily -> "Times", FontSize -> 14}];
A = Graphics3D[{Arrowheads[Large],
      Arrow[{{0, 0, 0}, {0, 0, 6}}], Text["z", {0, 0, 7}]},
   Boxed -> False, AxesEdge -> {{-1, 0}, None, None}];
B = Graphics3D[{Arrowheads[Tiny],
      Arrow[{{0, 0, 0}, {-6, 0, 0}}], Text["x", {-4, 1, 0}],
    Arrow[{{0, 0, 0}, {0, -6, 0}}], Text["y", {0, -7, 0}]},
   Boxed -> False];
p2 = RegionPlot3D[
     x < -1 && y > 1 && z > 1, {x, -6, 0}, {y, 0, 6}, {z, 0, 6},
     Boxed -> False, PlotStyle -> Directive[Red, Opacity[0.5`]],
     Mesh -> None, AxesEdge -> {{-1, 0}, None, None}, Axes -> True,
     AxesLabel -> {"x", "y", "z"},  Ticks -> None];
p3 = RegionPlot3D[
      x < -1 && y < -1 && z > 1, {x, -6, 0}, {y, -6, 0}, {z, 0, 6},
      Boxed -> False, PlotStyle -> Directive[Pink, Opacity[0.3`]],
      Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
      AxesLabel -> {"x", "y", "z"},
      PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
X =Graphics3D[{
    Text[Style["X(d2,d2,d1)", FontSize -> 12], {-1, 0, 0}, {.5,
      1}]}];
Y =Graphics3D[{
    Text[Style["Y(d2,d2,d1)", FontSize -> 12], {0, -1, 0}, {-1,
      1.5}]}];
Z =Graphics3D[{
    Text[Style["Z(d2,d2,d1)", FontSize -> 12], {0, 0,
      1}, {-1, -1}]}];
pi1 =Graphics3D[{PointSize[0.02], Point[{-1, 0, 0}]}];
pi2 =Graphics3D[{PointSize[0.02], Point[{0, -1, 0}]}];
pi3 =Graphics3D[{PointSize[0.02], Point[ {0, 0, 1}]}];
Show[A, B, p3, X, Y, Z, pi1, pi2, pi3]
```

Code for Figure 3.5.

```
SetOptions[Plot,
  BaseStyle -> {FontFamily -> "Times", FontSize -> 14}];
A = Graphics3D[{Arrowheads[Large],  Arrow[{{0, 0, 0}, {6, 0, 0}}],
    Text["x", {7, 0, 0}], Arrow[{{0, 0, 0}, {0, 0, 6}}],
    Text["z", {0, 0, 7}] } , Boxed -> False,
   AxesEdge -> {{-1, 0}, None, None}];
 B = Graphics3D[{Arrowheads[Tiny],  Arrow[{{0, 0, 0}, { 0, -6, 0}}],
    Text["y", {0, 7, 0}]} , Boxed -> False];
 p4 = RegionPlot3D[
         x > 1 && y < -1 && z > 1, {x, 0, 6}, {y, -6, 0}, {z, 0, 6},
         Boxed -> False,
   PlotStyle -> Directive[Blue, Opacity[0.3`]],
         Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
         AxesLabel -> {"x", "y", "z"},
         PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];

 X = Graphics3D[{
    Text[Style["X(d1,d2,d1)", FontSize -> 12], {1, 0, 0}, {0, -1}]}];
 Y = Graphics3D[{
    Text[Style["Y(d1,d2,d1)", FontSize -> 12], {0, -1,
      0}, {0, -1}]}];
 Z = Graphics3D[{
    Text[Style["Z(d1,d2,d1)", FontSize -> 12], {0, 0, 1}, {-1, 1}]}];
 pi1 = Graphics3D[{PointSize[0.02], Point[{1, 0, 0}]}];
 pi2 = Graphics3D[{PointSize[0.02], Point[{0, -1, 0}]}];
 pi3 = Graphics3D[{PointSize[0.02], Point[ {0, 0, 1}]}];

 Show[A, B, p4, X, Y, Z, pi1, pi2, pi3]
```

Code for Figure 3.6.

```
 SetOptions[Plot,
  BaseStyle -> {FontFamily -> "Times", FontSize -> 14}];
A = Graphics3D[{Arrowheads[Large],  Arrow[{{0, 0, 0}, {6, 0, 0}}],
    Text["x", {7, 0, 0}], Arrow[{{0, 0, 0}, {0, 6, 0}}],
    Text["y", {0, 7, 0}]} , Boxed -> False,
   AxesEdge -> {{-1, 0}, None, None}];
 B = Graphics3D[{Arrowheads[Tiny],
        Arrow[{{0, 0, 0}, {0, 0, - 6}}], Text["z", {0, 0, -7}]} ,
   Boxed -> False];

 p5 = RegionPlot3D[
         x > 1 && y > 1 && z < - 1, {x,  0, 6}, {y, 0, 6}, {z, -6,
    0},
         Boxed -> False,
   PlotStyle -> Directive[Green, Opacity[0.3`]],
         Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
         AxesLabel -> {"x", "y", "z"},
         PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];

X =Graphics3D[{
    Text[Style["X(d1,d2,d2)", FontSize -> 12], {1, 0, 0}, {0, -1}]}];
Y =Graphics3D[{
    Text[Style["Y(d1,d2,d2)", FontSize -> 12], {0, 1, 0}, {0, -1}]}];
 Z = Graphics3D[{
    Text[Style["Z(d1,d2,d2)", FontSize -> 12], {0, 0, -1}, {-1,
      1}]}];
pi1 = Graphics3D[{PointSize[0.02], Point[{1, 0, 0}]}];
 pi2 = Graphics3D[{PointSize[0.02], Point[{0, 1, 0}]}];
 pi3 = Graphics3D[{PointSize[0.02], Point[ {0, 0, -1}]}];

 Show[A, B, p5, X, Y, Z, pi1, pi2, pi3]
```

Code for Figure 3.7.

```
SetOptions[Plot,
  BaseStyle -> {FontFamily -> "Times", FontSize -> 14}];
A = Graphics3D[{Arrowheads[Large], Arrow[{{0, 0, 0}, {0, 6, 0}}],
    Text["y", {0, 7, 0}]}
    , Boxed -> False, AxesEdge -> {{-1, 0}, None, None}];
B = Graphics3D[{Arrowheads[Tiny],  Arrow[{{0, 0, 0}, {-6, 0, 0}}],
    Text["x", {-7, 0, 0}],
        Arrow[{{0, 0, 0}, {0, 0, - 6}}], Text["z", {0, 0, -7}]} ,
   Boxed -> False];
p6 = RegionPlot3D[
      x < -1 && y > 1 && z < - 1, {x, -6, 0}, {y, 0, 6}, {z, -6, 0},
      Boxed -> False, PlotStyle -> Directive[Brown, Opacity[0.5`]],
      Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
      AxesLabel -> {"x", "y", "z"},
      PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
X =Graphics3D[{
    Text[Style["X(d2,d1,d2)", FontSize -> 12], {-1, 0, 0}, {-1,
      1.5}]}];
Y =Graphics3D[{
    Text[Style["Y(d2,d1,d2)", FontSize -> 12], {0, 1,
      0}, {-0.5, -1.5}]}];
Z =Graphics3D[{
    Text[Style["Z(d2,d1,d2)", FontSize -> 12], {0, 0, -1}, {-1,
      1}]}];
pi1 =Graphics3D[{PointSize[0.02], Point[{-1, 0, 0}]}];
pi2 =Graphics3D[{PointSize[0.02], Point[{0, 1, 0}]}];
pi3 =Graphics3D[{PointSize[0.02], Point[ {0, 0, -1}]}];
Show[A, B, p6, X, Y, Z, pi1, pi2, pi3]
```

Code for Figure 3.8.

```
SetOptions[Plot,
  BaseStyle -> {FontFamily -> "Times", FontSize -> 14}];
A = Graphics3D[{Arrowheads[Large], Arrow[{{0, 0, 0}, {0, 6, 0}}],
    Text["y", {0, 7, 0}]}
    , Boxed -> False, AxesEdge -> {{-1, 0}, None, None}];
B = Graphics3D[{Arrowheads[Tiny],  Arrow[{{0, 0, 0}, {-6, 0, 0}}],
    Text["x", {-7, 0, 0}], Arrow[{{0, 0, 0}, {0, -6, 0}}],
    Text["y", {0, -7, 0}] ,
        Arrow[{{0, 0, 0}, {0, 0, - 6}}], Text["z", {0, 0, -7}]} ,
   Boxed -> False];
p7 = RegionPlot3D[
      x < -1 && y < -1 && z < - 1, {x, -6, 0}, {y, -6, 0}, {z, -6,
    0},
      Boxed -> False,
   PlotStyle -> Directive[Magenta, Opacity[0.5`]],
      Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
      AxesLabel -> {"x", "y", "z"},
      PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
X =Graphics3D[{
    Text[Style["X(d2,d2,d2)", FontSize -> 12], {-1, 0,
      0}, {0, -1}]}];
Y =Graphics3D[{
    Text[Style["Y(d2,d2,d2)", FontSize -> 12], {0, -1,
      0}, {0, -1}]}];
Z =Graphics3D[{
    Text[Style["Z(d2,d2,d2)", FontSize -> 12], {0, 0, -1}, {-1,
      1}]}];
pi1 =Graphics3D[{PointSize[0.02], Point[{-1, 0, 0}]}];
pi2 =Graphics3D[{PointSize[0.02], Point[{0, -1, 0}]}];
pi3 =Graphics3D[{PointSize[0.02], Point[ {0, 0, -1}]}];
Show[B, p7, X, Y, Z, pi1, pi2, pi3]
```

Code for Figure 3.9.

```
SetOptions[Plot,
  BaseStyle -> {FontFamily -> "Times", FontSize -> 14}];
A = Graphics3D[{Arrowheads[Large],  Arrow[{{0, 0, 0}, {6, 0, 0}}],
    Text["x", {7, 0, 0}]}
    , Boxed -> False, AxesEdge -> {{-1, 0}, None, None}];
B = Graphics3D[{Arrowheads[Tiny], Arrow[{{0, 0, 0}, {0, -6, 0}}],
    Text["y", {0, -7, 0}] ,
        Arrow[{{0, 0, 0}, {0, 0, - 6}}], Text["z", {0, 0, -7}]} ,
   Boxed -> False];
p8 = RegionPlot3D[
      x > 1 && y < -1 && z < - 1, {x, 0, 6}, {y, -6, 0}, {z, -6, 0},
      Boxed -> False, PlotStyle -> Directive[Cyan, Opacity[0.5`]],
      Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
      AxesLabel -> {"x", "y", "z"},
      PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
X =Graphics3D[{
    Text[Style["X(d1,d2,d2)", FontSize -> 12], {1, 0, 0}, {0, -1}]}];
Y =Graphics3D[{
    Text[Style["Y(d1,d2,d2)", FontSize -> 12], {0, -1,
      0}, {0, -1}]}];
Z =Graphics3D[{
    Text[Style["Z(d1,d2,d2)", FontSize -> 12], {0, 0, -1}, {-1,
      1}]}];
pi1 =Graphics3D[{PointSize[0.02], Point[{1, 0, 0}]}];
pi2 =Graphics3D[{PointSize[0.02], Point[{0, -1, 0}]}];
pi3 =Graphics3D[{PointSize[0.02], Point[ {0, 0, -1}]}];
Show[A, B, p8, X, Y, Z, pi1, pi2, pi3]
```

Code for Figure 3.10.

```
p1 = RegionPlot3D[
        x > 1 && y > 1 && z > 1, {x, -6, 6}, {y, -6, 6}, {z, -6, 6},
        Boxed -> False,
   PlotStyle -> Directive[Yellow, Opacity[0.5`]],
        Mesh -> None, AxesEdge -> {{0, 0}, {0, 0}, {0, 0}},
      Axes -> True,
        AxesLabel -> {"x", "y", "z"},
        PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
p2 = RegionPlot3D[ x < -1 && y > 1 && z > 1, {x, -6, 6}, {y, -6, 6}, {z, -6, 6},
       Boxed -> False, PlotStyle -> Directive[Red, Opacity[0.5`]],
       Mesh -> None, AxesEdge -> {{0, 0}, {0, 0}, {0, 0}},  Axes -> True,
       AxesLabel -> {"x", "y", "z"},
       PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
p3 = RegionPlot3D[
        x > 1 && y < -1 && z > 1, {x, -6, 6}, {y, -6, 6}, {z, -6, 6},
        Boxed -> False,
   PlotStyle -> Directive[Blue, Opacity[0.5`]],
        Mesh -> None, AxesEdge -> {{0, 0}, {0, 0}, {0, 0}},  Axes -> True,
        AxesLabel -> {"x", "y", "z"},  PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}},
        Ticks -> None];
p4 = RegionPlot3D[
        x > 1 && y > 1 && z < -1, {x, -6, 6}, {y, -6, 6}, {z, -6, 6},
        Boxed -> False,
   PlotStyle -> Directive[Green, Opacity[0.5`]],
        Mesh -> None, AxesEdge -> {{0, 0}, {0, 0}, {0, 0}},  Axes -> True,
        AxesLabel -> {"x", "y", "z"},
        PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
p5 = RegionPlot3D[
        x < -1 && y < -1 && z > 1, {x, -6, 6}, {y, -6, 6}, {z, -6,
    6},
        Boxed -> False,
   PlotStyle -> Directive[Pink, Opacity[0.5`]],
        Mesh -> None, AxesEdge -> {{0, 0}, {0, 0}, {0, 0}},
      Axes -> True,
        AxesLabel -> {"x", "y", "z"},
        PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
p6 = RegionPlot3D[
        x < -1 && y > 1 && z < -1, {x, -6, 6}, {y, -6, 6}, {z, -6, 6},
        Boxed -> False,
   PlotStyle -> Directive[Brown, Opacity[0.5`]],
        Mesh -> None, AxesEdge -> {{0, 0}, {0, 0}, {0, 0}},
      Axes -> True,
        AxesLabel -> {"x", "y", "z"},
        PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
p7 = RegionPlot3D[
        x > 1 && y < -1 && z < -1, {x, -6, 6}, {y, -6, 6}, {z, -6,   6},
```

```
         Boxed -> False,
   PlotStyle -> Directive[Cyan, Opacity[0.5‘]],
         Mesh -> None, AxesEdge -> {{0, 0}, {0, 0}, {0, 0}},
      Axes -> True,
         AxesLabel -> {"x", "y", "z"},
         PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
p8 = RegionPlot3D[
         x < -1 && y < -1 && z < -1, {x, -6, 6}, {y, -6, 6}, {z, -6, 6},
         Boxed -> False,
      PlotStyle -> Directive[Magenta, Opacity[0.5‘]],
         Mesh -> None, AxesEdge -> {{0, 0}, {0, 0}, {0, 0}},
      Axes -> True,
         AxesLabel -> {"x", "y", "z"},
         PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
p9 = Show[{Plot3D[z = 0, {x, -5, 5}, {y, -5, 5}, Mesh -> None,
                AxesOrigin -> {0, 0, 0}, PlotRange -> {-6, 6}],
          Graphics3D[{Text["x", Scaled[{-.05, .5, 0}], {0, -1}],
                Text["y", Scaled[{.5, -.05, 0}], {0, -1}],
                Text["z", Scaled[{.5, .5, 1.1}]]}]}, Boxed -> False];
points = Graphics3D[
      {PointSize[0.00], Point[{0, 0, 0}]},
       Boxed -> False, Axes -> True,
   AxesLabel -> (Style[#, 16] &) /@ {"x", "y", "z"},
   AxesOrigin -> {0, 0, 0}, AxesStyle -> Arrowheads[10],
   Ticks -> None,
       PlotRange -> 5
    ];
Show[points, p1, p2, p3, p4, p5, p6, p7, p8]
```

Code for Figure 3.11.

```
SetOptions[Plot,
  BaseStyle -> {FontFamily -> "Times", FontSize -> 14}];
A = Graphics3D[{Arrowheads[Large],  Arrow[{{0, 0, 0}, {8, 0, 0}}],
    Text["x", {9, 0, 0}], Arrow[{{0, 0, 0}, {0, 8, 0}}],
    Text["y", {0, 9, 0}] ,
        Arrow[{{0, 0, 0}, {0, 0, 8}}], Text["z", {0, 0, 9}]} ,
   Boxed -> False, AxesEdge -> {{-1, 0}, None, None}];
B = Graphics3D[{Arrowheads[Tiny],  Arrow[{{0, 0, 0}, {-9, 0, 0}}],
    Arrow[{{0, 0, 0}, {0, -9, 0}}] ,
        Arrow[{{0, 0, 0}, {0, 0, - 9}}]} , Boxed -> False];
p1 = RegionPlot3D[
         x > 1 && y > 1 && z > 1, {x, 0, 6}, {y, 0, 6}, {z, 0,
        6},    Boxed -> False,
      PlotStyle -> Directive[Yellow, Opacity[0.5`]],
         Mesh -> None, AxesEdge -> {{0, 0}, {0, 0}, {0, 0}},
      AxesLabel -> {Style["x", FontSize -> 16, FontWeight -> Bold],
          Style["y", FontSize -> 16, FontWeight -> Bold],
          Style["z", FontSize -> 16, FontWeight -> Bold]},
      PlotRange -> {{-6, 5}, {-6, 5}, {-6, 5}},    Ticks -> None];
p2 = RegionPlot3D[
        x < -1 && y > 1 && z > 1, {x, -6, 0}, {y, 0, 6}, {z, 0, 6},
        Boxed -> False, PlotStyle -> Directive[Red, Opacity[0.5`]],
        Mesh -> None, AxesEdge -> {{-1, 0}, None, None},
   Axes -> True,
        AxesLabel -> {"x", "y", "z"},  Ticks -> None];
p3 = RegionPlot3D[
         x < -1 && y < -1 && z > 1, {x, -6, 0}, {y, -6, 0}, {z, 0,
    6},
         Boxed -> False,
   PlotStyle -> Directive[Pink, Opacity[0.3`]],
         Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
         AxesLabel -> {"x", "y", "z"},
         PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
p4 = RegionPlot3D[
         x > 1 && y < -1 && z > 1, {x, 0, 6}, {y, -6, 0}, {z, 0, 6},
         Boxed -> False,
   PlotStyle -> Directive[Blue, Opacity[0.3`]],
         Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
         AxesLabel -> {"x", "y", "z"},
         PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
p5 = RegionPlot3D[
         x > 1 && y > 1 && z < - 1, {x,  0, 6}, {y, 0, 6}, {z, -6,
    0},
         Boxed -> False,
   PlotStyle -> Directive[Green, Opacity[0.3`]],
         Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
```

```
         AxesLabel -> {"x", "y", "z"},
         PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
p6 = RegionPlot3D[
         x < -1 && y > 1 && z < - 1, {x, -6, 0}, {y, 0, 6}, {z, -6,
    0},
         Boxed -> False,
   PlotStyle -> Directive[Brown, Opacity[0.5`]],
         Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
         AxesLabel -> {"x", "y", "z"},
         PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
p7 = RegionPlot3D[
         x < -1 && y < -1 && z < - 1, {x, -6, 0}, {y, -6,
    0}, {z, -6,
        0},
         Boxed -> False,
      PlotStyle -> Directive[Magenta, Opacity[0.5`]],
         Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
         AxesLabel -> {"x", "y", "z"},
         PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
p8 = RegionPlot3D[
      x > 1 && y < -1 && z < - 1, {x, 0, 6}, {y, -6, 0}, {z, -6, 0},
      Boxed -> False, PlotStyle -> Directive[Cyan, Opacity[0.5`]],
      Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
      AxesLabel -> {"x", "y", "z"},
      PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
X =Graphics3D[{
    Text[Style["X(d1,d2,d2)", FontSize -> 12], {1, 0, 0}, {0, -1}]}];
Y =Graphics3D[{
    Text[Style["Y(d1,d2,d2)", FontSize -> 12], {0, -1,
      0}, {0, -1}]}];
Z =Graphics3D[{
    Text[Style["Z(d1,d2,d2)", FontSize -> 12], {0, 0, -1}, {-1,
      1}]}];
pi1 =Graphics3D[{PointSize[0.02], Point[{1, 0, 0}]}];
pi2 =Graphics3D[{PointSize[0.02], Point[{0, -1, 0}]}];
pi3 =Graphics3D[{PointSize[0.02], Point[ {0, 0, -1}]}];
Show[A, B, p1, p2, p3, p4, p5, p6, p7, p8]
```

Code for Figure 3.12.

```
SetOptions[Plot,
  BaseStyle -> {FontFamily -> "Times", FontSize -> 14}];
A = Graphics3D[{Arrowheads[Large],  Arrow[{{0, 0, 0}, {8, 0, 0}}],
   Text["x", {9, 0, 0}], Arrow[{{0, 0, 0}, {0, 8, 0}}],
   Text["y", {0, 9, 0}] ,
       Arrow[{{0, 0, 0}, {0, 0, 8}}], Text["z", {0, 0, 9}]} ,
  Boxed -> False, AxesEdge -> {{-1, 0}, None, None}];
B = Graphics3D[{Arrowheads[Tiny],  Arrow[{{0, 0, 0}, {-9, 0, 0}}],
   Arrow[{{0, 0, 0}, {0, -9, 0}}] ,
       Arrow[{{0, 0, 0}, {0, 0, - 9}}]} , Boxed -> False];
p1 = RegionPlot3D[
         x > -3 && y > -3 && z > -3, {x, -6, 6}, {y, -6, 6}, {z, -6,
    6},    Boxed -> False,
      PlotStyle -> Directive[Yellow, Opacity[0.5`]],
         Mesh -> None,    Ticks -> None];
p2 = RegionPlot3D[
        x < 3 && y > -3 && z > -3, {x, -6, 6}, {y, -6, 6}, {z, -6,
    6},
        Boxed -> False, PlotStyle -> Directive[Red, Opacity[0.5`]],
        Mesh -> None, AxesEdge -> {{-1, 0}, None, None},
   Axes -> True,
        AxesLabel -> {"x", "y", "z"},  Ticks -> None];
p3 = RegionPlot3D[
         x < 3 && y < 3 && z > -3, {x, -6, 6}, {y, -6, 6}, {z, -6,
    6},
         Boxed -> False,
   PlotStyle -> Directive[Pink, Opacity[0.3`]],
         Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
         AxesLabel -> {"x", "y", "z"},
         PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
p4 = RegionPlot3D[
         x > -3 && y < 3 && z > -3, {x, -6, 6}, {y, -6, 6}, {z, -6,
    6},
         Boxed -> False,
   PlotStyle -> Directive[Blue, Opacity[0.3`]],
         Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
         AxesLabel -> {"x", "y", "z"},
         PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
p5 = RegionPlot3D[
         x > -3 && y > -3 && z < 3, {x, -6, 6}, {y, -6, 6}, {z, -6,
    6},
         Boxed -> False,
   PlotStyle -> Directive[Green, Opacity[0.3`]],
         Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
         AxesLabel -> {"x", "y", "z"},
         PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
```

```
p6 = RegionPlot3D[
         x < 3 && y > -3 && z < 3, {x, -6, 6}, {y, -6, 6}, {z, -6,
    6},
         Boxed -> False,
   PlotStyle -> Directive[Brown, Opacity[0.5`]],
         Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
         AxesLabel -> {"x", "y", "z"},
         PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
p7 = RegionPlot3D[
         x < 3 && y < 3 && z < 3, {x, -6, 6}, {y, -6, 6}, {z, -6,
    6},
         Boxed -> False,
      PlotStyle -> Directive[Magenta, Opacity[0.5`]],
         Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
         AxesLabel -> {"x", "y", "z"},
         PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
p8 = RegionPlot3D[
      x > -3 && y < 3 && z < 3, {x, -6, 6}, {y, -6, 6}, {z, -6, 6},
      Boxed -> False, PlotStyle -> Directive[Cyan, Opacity[0.5`]],
      Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
      AxesLabel -> {"x", "y", "z"},
      PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
X =Graphics3D[{
    Text[Style["X(d1,d2,d2)", FontSize -> 12], {1, 0, 0}, {0, -1}]}];
Y =Graphics3D[{
    Text[Style["Y(d1,d2,d2)", FontSize -> 12], {0, -1,
      0}, {0, -1}]}];
Z =Graphics3D[{
    Text[Style["Z(d1,d2,d2)", FontSize -> 12], {0, 0, -1}, {-1,
      1}]}];
pi1 =Graphics3D[{PointSize[0.02], Point[{1, 0, 0}]}];
pi2 =Graphics3D[{PointSize[0.02], Point[{0, -1, 0}]}];
pi3 =Graphics3D[{PointSize[0.02], Point[ {0, 0, -1}]}];
Show[A, B, p1, p2, p3, p4, p5, p6, p7, p8]
```

Code for Figure 3.13.

```
SetOptions[Plot,  BaseStyle -> {FontFamily -> "Times", FontSize -> 14}];
A=Graphics3D[{Arrowheads[Large], Arrow[{{0,0, 0}, {10, 0, 0}}],Text["x",{11,0,0}],
    Arrow[{{0,0, 0}, {0,10,0}}], Text["y",{0,11,0}], Arrow[{{0,0, 0}, {0, 0,10}}],
    Text["z",{0,0,11}]} ,Boxed->False, AxesEdge -> {{-1,0},None,None}];
  B=Graphics3D[{Arrowheads[Tiny],  Arrow[{{0,0, 0}, {-10, 0, 0}}],
  Arrow[{{0,0, 0}, {0,-10,0}}], Arrow[{{0,0, 0}, {0, 0,- 10}}]} ,Boxed->False];
    p1 = RegionPlot3D[
      x > -4 && y > -2 && z > -4, {x, -10, 10}, {y, -10, 10}, {z,-10, 10},
      Boxed -> False, PlotStyle -> Directive[Yellow, Opacity[0.5`]],
      Mesh -> None,    Ticks -> None];
   p2 = RegionPlot3D[
     x < 0 && y > 0 && z >0,{x, -10, 10}, {y, -10, 10}, {z, -10, 10},
     Boxed -> False, PlotStyle -> Directive[Red, Opacity[0.5`]],
     Mesh -> None, AxesEdge -> {{-1, 0}, None, None}, Axes -> True,
     AxesLabel -> {"x", "y", "z"},  Ticks -> None];
     p3 = RegionPlot3D[
      x <0 && y < 6 && z > 0, {x,-10, 10}, {y, -10, 10}, {z,-10, 10},
      Boxed -> False, PlotStyle -> Directive[Pink, Opacity[0.3`]],
      Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
      AxesLabel -> {"x", "y", "z"},
      PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
       p4= RegionPlot3D[
      x >0 && y < 0 && z > 0, {x, -10, 10}, {y, -10, 10}, {z, -10, 10},
      Boxed -> False, PlotStyle -> Directive[Blue, Opacity[0.3`]],
      Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
      AxesLabel -> {"x", "y", "z"},
      PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
      p5 = RegionPlot3D[
      x >-3 && y >0 && z < 3, {x, -10, 10}, {y,-10, 10}, {z, -10, 10},
      Boxed -> False, PlotStyle -> Directive[Green, Opacity[0.3`]],
      Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
      AxesLabel -> {"x", "y", "z"},
      PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
      p6 = RegionPlot3D[
      x <0 && y > 0 && z <0, {x,-10, 10}, {y,-10, 10}, {z, -10, 10},
      Boxed -> False, PlotStyle -> Directive[Brown, Opacity[0.5`]],
      Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
      AxesLabel -> {"x", "y", "z"},
      PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
      p7 = RegionPlot3D[
      x < 0 && y <0 && z < 0, {x, -10, 10}, {y,-10, 10}, {z, -10, 10},
      Boxed -> False,
   PlotStyle -> Directive[Magenta, Opacity[0.5`]],
      Mesh -> None, AxesEdge -> {None, None, None}, Axes -> True,
      AxesLabel -> {"x", "y", "z"},
      PlotRange -> {{-5, 5}, {-5, 5}, {-5, 5}}, Ticks -> None];
```

```
  p8 = RegionPlot3D[
x >-4 && y <0 && z <4, {x, -10, 10}, {y, -10, 10}, {z,-10, 10},
Boxed -> False, PlotStyle -> Directive[Cyan, Opacity[0.5`]],
Mesh -> None, AxesEdge -> {None,None,None}, Axes -> True,
AxesLabel -> {"x", "y", "z"},
Ticks -> None];

Show[A,B,p1,p2,p3,p4,p5,p6,p7,p8]
```

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